

Persuasion with Limited Communication

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June 28, 2016

Abstract

We examine information structure design, also called “persuasion” or “signaling,” in the presence of a constraint on the amount of communication. We focus on the fundamental setting of bilateral trade, which in its simplest form involves a seller with a single item to price, a buyer whose value for the item is drawn from a common prior distribution over n different possible values, and a take-it-or-leave-it-offer protocol. A mediator with access to the buyer’s type may partially reveal such information to the seller in order to further some objective such as the social welfare or the seller’s revenue. We study how a limit on the number of bits of communication affects this setting in two respects: (1) How much does this constraint reduce the optimal welfare or revenue? (2) What effect does constraining communication have on the computational complexity of the mediator’s optimization problem?

In the setting of maximizing welfare under bilateral trade, we exhibit positive answers for both questions (1) and (2). Whereas the optimal unconstrained scheme may involve n signals (and thus $\log(n)$ bits of communication), we show that $O(\log(n) \log \frac{1}{\epsilon})$ signals suffice for a $1 - \epsilon$ approximation to the optimal welfare, and this bound is tight. This largely justifies the design of algorithms for signaling subject to drastic limits on communication. As our main result, we exhibit an efficient algorithm for computing a $\frac{M-1}{M} \cdot (1 - 1/e)$ -approximation to the welfare-maximizing scheme with at most M signals. This result hinges on an intricate submodularity argument which relies on the optimality of a greedy algorithm for solving a certain linear program. For the revenue objective, the surprising logarithmic bound on the number of signals does not carry over: we show that $\Omega(n)$ signals are needed for a constant factor approximation to the revenue of a fully informed seller. From a computational perspective, however, the problem gets easier: we show that a simple dynamic program computes the signaling scheme with M signals maximizing the seller’s revenue.

Observing that the signaling problem in bilateral trade is a special case of the fundamental *Bayesian Persuasion* model of Kamenica and Gentzkow, we also examine the question of communication-constrained signaling more generally. Specifically, in this model there is a sender (the mediator), a receiver (the seller) looking to take an action (setting the price), and a state of nature (the buyer’s type) drawn from a common prior. The state of nature encodes both the receiver’s utility and the sender’s objective as a function of the receiver’s action. Our results for bilateral trade with the revenue objective imply that limiting communication to M signals can scale the sender’s utility by a factor of $O(\frac{M}{n})$ in general, where n denotes the number of states of nature. We also show that our positive algorithmic results for bilateral trade do not extend to communication-constrained signaling in the Bayesian Persuasion model. Specifically, we show that it is NP-hard to approximate the optimal sender’s utility to within any constant factor in the presence of communication constraints.

1 Introduction

Strategic interactions are often rife with uncertainty and information asymmetry. Auctions and markets on the Internet feature sellers with privileged information regarding their products, and buyers with private information regarding their willingness to pay. The *information structures* which govern these interactions play a key role in determining the equilibria of the resulting games. In Akerlof’s *market for lemons* [1], for instance, information asymmetries between the buyers and sellers of used cars — buyers cannot distinguish good cars from bad whereas sellers can — can lead to the collapse of the entire market. In other cases, information asymmetries can serve a useful purpose; for example, a seller of advertising impressions may conflate different impressions in order to prevent “cherry picking” by advertisers, increase competition, and as a result increase her¹ revenue [21]. It is for these reasons that information structure design, also known as *signaling*, is emerging as a new *mechanism design for information*. This new frontier, like traditional mechanism design, raises deep algorithmic and complexity-theoretic questions whose exploration has only recently begun (see, e.g., [6, 9, 11–13, 16]).

Perhaps one of the most fundamental economic interactions governed by the presence or absence of information is *bilateral trade* between two parties [23]. In (a simplified form of) bilateral trade, one side can choose whether to participate in the trade, and by doing so would generate a social surplus which is private information to him. The other side can propose to take a fixed amount of the social surplus. Two particularly natural instantiations of bilateral trade are the following:

1. Trade of an item between a seller (who has no value for the item) and a buyer via a posted price. The seller’s posted price is the amount of surplus she proposes to take, while the buyer’s valuation for the item, drawn from a commonly known distribution, is the amount of social surplus the trade would generate. The buyer chooses whether to accept the seller’s posted price. We call the resulting game the *pricing game*.
2. Trade between an employer and an agent:² an employer would like to hire an agent to complete a project, and has (known) utility u for its completion. The agent has a private cost c (drawn from a known distribution) for completing the project, and the social surplus generated is the difference $u - c$. Without knowing the cost of the agent, the employer posts a proposed payment p , which is equivalent to posting the share $u - p$ of the social welfare which she proposes to keep. We call this game the *employment game*.

The buyer/agent is assumed to be rational with quasilinear utility, while the seller/employer aims to maximize her own utility. The buyer/agent accepts an offer if he would derive non-negative utility from it, and rejects it otherwise; this corresponds to the valuation exceeding the price in the pricing game, and the payment exceeding the cost in the employment game. For concreteness, we will state all of our results in the language of the pricing game; however, all results carry over verbatim to the employment game, and we will occasionally remark on the interpretation of results in this context.

The seller’s chosen offer price to the buyer depends on what she knows about the buyer’s value. If the seller has no information other than the distribution Γ of values in the population of potential buyers, she chooses the price $P^* = P^*(\Gamma)$ maximizing her revenue $\text{Rev}(P, \Gamma) = P \cdot \Pr_{V \sim \Gamma}[V \geq P]$.

¹For clarity, we will throughout use female pronouns for the seller and male pronouns for the buyer.

²In the literature, this falls into the class of *principal-agent* models. However, in this article, we use the word “principal” for a different role, so we will use the non-standard nomenclature to avoid misunderstandings.

This leads to a revenue of $\text{Rev}(\Gamma) = \text{Rev}(P^*(\Gamma), \Gamma) = P^* \cdot \Pr_{V \sim \Gamma}[V \geq P^*]$ for the seller, and a social welfare of $\text{Welfare}(\Gamma) = \Pr_{\Gamma}[V \geq P^*] \cdot \mathbb{E}_{V \sim \Gamma}[V \mid V \geq P^*]$ for both players combined. At the other extreme is the case when the seller is fully informed about V . She can now set a price $P = V$; trade will always occur, leading to a maximum social welfare of $\mathbb{E}_{V \sim \Gamma}[V]$, which is fully extracted as revenue by the seller. Notice the difference caused by different amounts of information being communicated: with no information, the social welfare can be arbitrarily smaller than with full information.

In order to fully inform the seller, very fine-grained information had to be communicated. In reality, for practical and logistical reasons, the information received by the seller about the buyer is typically limited. For example, in the employment game, the information may be provided by a university or certification agency, which may initially only be able to communicate a coarse-grained rating of the agent via a GPA or the performance on a certification test. When the “agent” provides a product (e.g., a piece of clothing or medication), the location or label it is sold under (boutique/brand-name vs. discount/generic) sends a coarse signal to the potential employer/buyer about the distribution of qualities she is to expect. Indeed, the study of communication constraints and their impact on the outcomes of games dates back at least to the work of [4, 5] on communication constraints in auctions.

*How to optimally inform the seller with limited communication is the subject of the present article.*³ This question actually comprises two separate thrusts: (1) What is the inherent *price of limited communication*? In other words, how much social welfare is lost because the principal can only communicate limited information to the seller? A particularly stark version of this question is the following: in the presence of a self-interested seller, how much social welfare can a principal salvage by sending a single bit of information, which is equivalent to merely being able to *exclude* some buyers from the market? (2) What are the *algorithmic* consequences of limited communication? How well can the principal optimize the welfare with limited communication, if the computation has to be efficient as well?

In order to formalize the notion of limited communication, we first define information structures. At its most general, an *information structure* for the seller is a (possibly randomized) map φ from the realized value $V \in \mathbb{R}$ of the buyer to a signal $\sigma \in \Sigma$ presented to the seller. This in effect partitions the probability histogram of Γ —or equivalently, the population of buyers—into different segments, with each corresponding to a signal. Specifically, we write $\varphi(v, \sigma) = \Pr[\varphi(v) = \sigma \mid V = v]$, where the randomness is over the internal coins of φ . Receiving signal σ induces, via Bayes’ rule, a posterior distribution μ_σ for the seller, characterized by $\Pr_{\mu_\sigma}[V = v] = \frac{\Pr_{\Gamma}[V=v] \cdot \varphi(v, \sigma)}{\Pr[\varphi(V) = \sigma]}$; here, the randomness in the denominator is over both Γ and the internal coins of φ . Upon receiving σ , the seller’s optimal price is $P^*(\mu_\sigma)$ maximizing $\text{Rev}(P, \mu_\sigma)$. This price induces a revenue of $\text{Rev}(\mu_\sigma)$ for the seller and a social welfare of $\text{Welfare}(\mu_\sigma)$. The expected revenue and social welfare over all draws of the buyer’s value and randomness in the scheme φ are then given by $\text{Rev}(\varphi, \Gamma) = \sum_{\sigma \in \Sigma} \Pr[\varphi(V) = \sigma] \cdot \text{Rev}(\mu_\sigma)$ and $\text{Welfare}(\varphi, \Gamma) = \sum_{\sigma \in \Sigma} \Pr[\varphi(V) = \sigma] \cdot \text{Welfare}(\mu_\sigma)$, respectively.

We now revisit our motivating examples. Suppose that the distribution Γ has support $\{v_1, v_2, \dots, v_n\}$. Then, V can be precisely communicated to the seller by choosing a signal set $\Sigma = \{1, \dots, n\}$ and setting $\varphi(v_i, i) = 1$ (and $\varphi(v_i, j) = 0$ for $i \neq j$). By way of contrast, the signaling scheme communicating no information to the seller is implemented with $\Sigma = \{1\}$ and $\varphi(v_i, 1) = 1$ for all i . Notice that the latter uses much lower “communication complexity,” as measured by $|\Sigma|$. Also notice

³ We note that an alternate interpretation of this goal is in terms of *market segmentation*. Specifically, *how would a market designer optimally partition the market into a limited number of segments?*

that both these extreme information structures are inherently algorithmically efficient: given an explicit representation of Γ and a value V , it is trivial to compute $\varphi(V)$. For the design of signaling schemes, we use the constraint that $|\Sigma| = M$, for a given bound M , as the “low communication complexity” constraint.

We would like to understand the impact of the communication complexity on the social welfare that can be (in principle) achieved, and to analyze the algorithmic question of computing optimal signaling schemes φ with limited communication complexity. The “gold standard” for social welfare is $\mathbb{E}_{V \sim \Gamma} [V]$, which we will call the full-information welfare.

Formally, we are interested in the following questions: Given an explicit representation of the distribution Γ and a bound M on the number of available signals, (1) What fraction of the full-information welfare can be obtained by a signaling scheme with M signals? (2) Are there (approximately) optimal and computationally efficient signaling schemes for maximizing social welfare?

Our Results

Our first main result (proved in Section 3) shows that signaling schemes even with an extremely limited number of signals are surprisingly powerful in extracting welfare.

Theorem 1.1 *For any distribution Γ with support size n , there is a single segment of the market (i.e., a non-negative vector indexed by buyer types that is pointwise upper-bounded by the prior over buyer types) with social welfare at least a $\Theta(1/\log n)$ fraction of the full-information welfare $\mathbb{E}_{V \sim \Gamma} [V]$. The $\Theta(1/\log n)$ bound is tight.*

Notice that this result is quite surprising. There are value distributions (such as equal-revenue distributions) under which the presence of a self-interested seller results in only a fraction $1/n$ of the full-information welfare being realized. The theorem states that merely by *excluding* some buyers, the principal can improve this bound to $\Theta(1/\log n)$. Perhaps a “natural” conjecture would have been that using M segments, at most a fraction $O(M/n)$ of welfare could be attained in the worst case. The theorem and the subsequent corollary show that this conjecture is false. However, as we will see shortly, it is in fact true for the objective of maximizing the seller’s *revenue*.

Applying Theorem 1.1 repeatedly yields Corollary 1.2, which shows that with a relatively small number of signals, we can get arbitrarily close to the full-information welfare.

Corollary 1.2 *For any $\epsilon > 0$ and any distribution Γ with support size n , there is a signaling scheme with $O(\log n \log(1/\epsilon))$ signals which obtains a $(1 - \epsilon)$ fraction of the full-information welfare.*

In terms of the number of bits of communication required, Corollary 1.2 implies that communicating $O(\log \log n + \log \log(1/\epsilon))$ bits extracts a $(1 - \epsilon)$ fraction of the social welfare that could be extracted using $\log n$ bits.

Corollary 1.2 also has the following algorithmic implication: exhaustively searching over all signaling schemes with $\Theta(\log n \cdot \log(1/\epsilon))$ signals, one obtains a QPTAS (quasi-polynomial time approximation scheme).

Obtaining a truly polynomial-time algorithm appears quite a bit more challenging, although — as discussed in Section 8 — we currently do not have any hardness results, even for exact optimization. Our main technical result is Theorem 1.3, which shows that one can obtain a constant factor approximation to the social welfare in polynomial time.

Theorem 1.3 *For any $M > 1$, there is a polynomial-time $\frac{M-1}{M} \cdot (1 - 1/e)$ approximation algorithm for the problem of implementing a welfare-maximizing signaling scheme with at most M signals, given an explicit representation of Γ .*

The proof of this theorem is quite involved. At the heart of it is a proof that the social welfare achieved from a set of signals is submodular. More precisely, the proof focuses on the social welfare accrued from all but one signal (which we denote by \perp and call the *garbage signal* — it is a signal of buyer types for which our solution will not be credited with any reward). Each non-garbage signal σ induces an optimal equilibrium price $P^*(\mu_\sigma)$ chosen by the seller. We can think of the computation of φ as first choosing the set $S = \{P_1, P_2, \dots, P_{M-1}\}$ of equilibrium prices of the optimum solution, and then choosing an optimal signaling scheme inducing these particular prices. The key insight is that the optimum social welfare with price set S (and one garbage signal) is a monotone and submodular function of S . This fact is proved as Theorem 5.1 in Section 5. The proof relies heavily on a characterization of the optimal signaling scheme inducing the set S of prices. While it is easy to see that, given the target prices $S = \{P_1, P_2, \dots, P_{M-1}\}$, this optimal signaling scheme can be computed using a linear program (see Section 2), we require a better characterization, and thereto show (as Theorem 4.3 in Section 4) that it is also the output of a greedy algorithm.

Optimizing Seller Revenue

The reader may have noticed our focus on social welfare. Almost equally frequently studied is the objective of maximizing seller revenue. It turns out that results for seller revenue are much more straightforward, both technically and in terms of their implications. First, the communication constraint can severely curtail the seller’s revenue: when the buyer’s value is drawn from an equal-revenue distribution supported on a geometric progression of length n , the revenue-maximizing signaling scheme with M signals recovers only an $O(M/n)$ fraction of the full-information revenue. On the other hand, *computing* the optimal signaling scheme for revenue maximization is straightforward (see Section 6):

Theorem 1.4 *The optimal signaling scheme for maximizing seller revenue groups buyers into M contiguous segments by valuations, i.e., if the same signal σ is sent for buyers with valuations $v < v'$, then σ is also sent for all buyers with valuations $v'' \in [v, v']$. As a result, there exists a polynomial-time dynamic programming algorithm which, given an explicit representation of Γ and a bound M on the number of signals, computes a signaling scheme maximizing the seller’s expected revenue.*

In the employment game, Theorem 1.4 confirms (for the purpose of employer utility maximization) the generally agreed-upon form of grading or performance evaluation, wherein the highest performers are grouped together in one category (‘A’), followed by the next highest category (‘B’), etc. Such signaling schemes are generally not optimal if the goal is to maximize social welfare, and indeed, the optimum signaling scheme for welfare maximization can be fairly complex. As an example, consider a buyer distribution supported on types $(1, 2, 3, 4)$ with probabilities $(2/12, 1/12, 2/12, 7/12)$, respectively. When $M = 2$ signals are allowed, the optimal signaling scheme obtains full welfare by sending signals with posterior unnormalized probabilities of $(2/12, 1/12, 0, 1/12)$ and $(0, 0, 2/12, 6/12)$.

Bayesian Persuasion

While our main focus is on bilateral trade, the framework of communication-bounded signaling naturally applies to other games as well. A more general setting has been termed *Bayesian Persuasion* by Kamenica and Gentzkow [18]: a sender observes a random variable capturing the “state of the world,” and can send a signal to a receiver. The receiver, based on the received signal, chooses an action. The utility of both the sender and the receiver depend on the state of the world and the chosen action, and are not necessarily aligned. Thus, the sender’s goal is to design the information structure such that the receiver will choose actions which are in expectation beneficial to him.

Notice that signaling in bilateral trade fits in this framework. In the pricing game, the seller is the receiver, the buyer’s valuation is the state of the world, and the sender is a market designer with the goal of maximizing the seller’s revenue or social welfare. In the employment game, the employer is the receiver, the employee’s cost is the state of the world, and the sender is an educational institution or crowdsourcing website aiming to generate welfare for its participants.

Our results for revenue in the pricing game already imply that the price of limited communication is high in some persuasion games. While it may be natural to conjecture that the submodularity property carries over from bilateral trade to more general persuasion games, this is not the case: we establish a strong hardness result for maximizing the sender’s utility in general. (A more formal version of this theorem and the proof are given in Section 7.)

Theorem 1.5 *For any constant $c > 0$, it is NP-hard to construct a signaling scheme approximating the maximum expected sender utility to within a factor c , given an explicit representation of a Bayesian Persuasion game and a bound M on the number of signals.*

Related Work

Our focus is on the classical model of *bilateral trade* (see [23, Chapter 23]). Our choice of protocol, namely the take-it-or-leave-it offer, is arguably the simplest mechanism for bilateral trade, and in the case of the pricing game corresponds to the revenue-optimal mechanism by the classical result of Myerson [24]. The study of the impact of auxiliary information on trade — also known as *third degree price discrimination* — has a long history, starting at least as early as [26]. We refer the reader to [3] for an in-depth look at this economic literature.

The work most directly related to ours is that of Bergemann et al. [3], who examine the effects of information in the same pricing game. Their main result is a remarkable characterization of buyer and seller expected utilities that are attainable by varying the information structure of the seller, i.e., by segmenting the market and allowing the seller to *price discriminate* between segments. They characterize the space of *realizable* pairs (r, u) for which there exists an information structure φ such that the seller’s expected revenue is r and the buyer’s expected utility is u : (r, u) is realizable if and only if $r \geq \text{Rev}(\Gamma)$ (i.e., the seller at least matches her “uninformed” revenue), $u \geq 0$, and $r + u \leq \mathbb{E}_{V \sim \Gamma}[V]$.

Implicit in [3] is a family of algorithms — parametrized by the distribution Γ and a realizable pair of utilities (r, u) — which *implement* a signaling scheme φ realizing the pair of utilities (r, u) . When Γ is an explicitly-described distribution with support size n , the signaling schemes implicit in [3] are efficient; their runtime is a low-order polynomial in n .⁴ However, the most interesting

⁴A particularly beautiful example of the schemes implicit in [3] is the greedy algorithm achieving $r = \text{Rev}(\Gamma)$ and $u = \mathbb{E}_{V \sim \Gamma}[V] - \text{Rev}(\Gamma)$.

signaling schemes implied in [3] — in particular those with largest and smallest u — use as many signals as the support size of the buyer distribution Γ . This realization motivates our examination of schemes with limited communication.

Roesler and Szentes [28] also study the impact of information revelation on bilateral trade. In their model, the buyer can observe a signal of his value for the item, and will pay for it if the conditional expected value is weakly larger than the price. The seller will choose the optimal monopoly price according to the buyer’s information structure.

The signaling problem in bilateral trade is a special case of *Bayesian Persuasion*, as formalized by Kamenica and Gentzkow [18], generalizing an earlier model by Brocas and Carrillo [7]. Instantiations, variants, and generalizations of the Bayesian Persuasion problem have seen a flurry of interest in recent years. For example, persuasion has been examined in the context of voting [2], security [27, 30], multi-armed bandits [20, 22], medical research [19], and financial regulation [14, 15]. Dughmi and Xu [12] also consider persuasion algorithmically. However, they focus on Bayesian Persuasion without any communication constraint, but allowing exponentially (or infinitely) many states of nature in the number of actions.

More generally, Bayesian Persuasion is a special case of optimal information structure design in games. Recent work in computer science has examined this question algorithmically, mostly in the context of auctions [6, 10, 11, 13, 16]. In all these works, the uncertainty (i.e., state of nature) concerns the item being sold, rather than the type of the buyer as in our model. Nevertheless, [10, 11] are related to our work in that they also examine communication-limited signaling schemes. The work of Dughmi [9] examines the complexity of signaling in abstract two-player normal form games, while the recent work of Cheng et al. [8] presents an algorithmic framework for tackling a number of (unconstrained) signaling problems.

An analogy can be drawn between our work and some of the work on auction design subject to communication constraints. Blumrosen et al. [5] study single-item auctions in which bidders can only communicate a limited number of bits to the auctioneer. They show that even severe bounds on communication only lead to mild losses in welfare and revenue. Moreover, they show that bidders simply report an interval in which their value for the item lies when faced with an optimal auction. Blumrosen and Feldman [4] study communication-constrained mechanism design in single-parameter problems more generally, and examine necessary and sufficient conditions under which a communication-constrained mechanism matches or approximates the optimal (unconstrained) mechanism.

2 Preliminaries

Throughout, we use the following conventions for notation. Vectors are denoted by bold face. When we write $\mathbf{x} \leq \mathbf{y}$ for vectors \mathbf{x}, \mathbf{y} , we mean that $x_i \leq y_i$ for all i . We will frequently want to reason about the sums of entries of a vector over a given set of indices. We then write $x_I = \sum_{i \in I} x_i$. We also apply this notation for elements of a matrix $X = (x_{i,j})_{i,j}$, writing $x_{I,J} = \sum_{i \in I} \sum_{j \in J} x_{i,j}$. We will particularly use this notation when I, J are (closed or half-open) intervals of integers.

2.1 Signaling Schemes

When constructing a signaling scheme, we assume that the distribution Γ of buyer valuations is given explicitly as input. In particular, this means that it must have finite support of size n . We

assume that it is given by the valuations $v_1 < v_2 < \dots < v_n$, and their associated probabilities p_1, p_2, \dots, p_n , satisfying $\sum_i p_i = 1$. We write \mathbf{v} and \mathbf{p} for the vectors of all these values and probabilities, respectively.

In the introduction, for ease of exposition, we described a signaling scheme φ in terms of its conditional probabilities $\varphi(v, \sigma) = \Pr[\varphi(v) = \sigma \mid V = v]$. For the remainder of this article, we use notation differing in two ways: (1) since all values are of the form v_i , we can index the buyer types by i instead of v , and (2) it is much more convenient to use unnormalized probabilities instead of conditional probabilities: $x_{i,\sigma} = \varphi(v_i, \sigma) \cdot \Pr[V = v_i]$ is the probability that the buyer's valuation is v_i and the signal σ is sent. The signaling scheme is then fully described by the matrix $X \in [0, 1]^{n \times M} = (x_{i,\sigma})_{i \in \{1, \dots, n\}, \sigma \in \{1, \dots, M\}}$, satisfying that $\sum_{\sigma} x_{i,\sigma} = p_i$. From now on, we will therefore simply refer to the signaling scheme as X instead of φ .

We sometimes describe a signal σ in isolation by a nonnegative type-indexed vector $\mathbf{0} \leq \mathbf{x}'_{\sigma} \leq \mathbf{p}$. We call such a vector a *segment* of the market \mathbf{p} . Thus, a signaling scheme can be thought of as a family of segments, one per signal, whose sum is the entire market \mathbf{p} .

As discussed in the introduction, upon receiving the signal σ , the seller will choose a price $P(\sigma)$ maximizing $P \cdot \Pr_{V \sim \mu_{\sigma}}[V \geq P]$. This price will always be one of the possible buyer valuations v_i , as any other price could be raised slightly without losing any buyers. Furthermore, by merging signals with the same price into one signal, without loss of generality, there are no two signals for which the seller chooses the same price [18]. Hence, any signaling scheme X induces indices k_1, k_2, \dots, k_M such that upon receiving signal σ , the seller chooses price $v_{k_{\sigma}}$. Without loss of generality, we can rearrange the signals so that $k_1 > k_2 > \dots > k_M$.

For any signal σ , the expected welfare resulting from σ under X is called the *served social welfare* and defined as $w_X(\sigma) = \sum_{i \geq k_{\sigma}} v_i \cdot x_{i,\sigma}$; the social welfare is then $W(X) = \sum_{\sigma} w_X(\sigma)$.

We call the signaling scheme X *optimal for welfare* if X maximizes $W(X)$. If $W(X) \geq \alpha W(X^*)$, where X^* is optimal for welfare, we call X an α -*approximation signaling scheme for welfare*.

2.2 Welfare and Sanitized Welfare

While our goal for most of this article is to maximize the social welfare, it turns out that a slightly modified objective function is significantly more amenable to analysis, both in terms of positive results for bilateral trade and the hardness result for more general persuasion. Specifically, there is a designated *garbage signal* \perp , and any welfare accrued when \perp is sent is discounted: we thus define the *sanitized welfare* of X to be $\widetilde{W}(X) = \sum_{\sigma \neq \perp} w_X(\sigma)$. By designating the signal σ minimizing $w_X(\sigma)$ as the garbage signal, we observe:

Proposition 2.1 *For all signaling schemes X , we have that $\frac{M-1}{M} \cdot W(X) \leq \widetilde{W}(X) \leq W(X)$.*

In particular, any signaling scheme X maximizing sanitized welfare to within a factor α also maximizes social welfare to within a factor $\frac{M-1}{M} \cdot \alpha$. Since we will always focus on sanitized welfare in the context of welfare maximization, to avoid having to write $M - 1$ for the number of signals everywhere, we will explicitly assume that our signaling schemes can use M signals *in addition to* the garbage signal \perp .

2.3 Sanitized Welfare Maximization Given Price Points

Conceptually, the task of designing a good signaling scheme can be divided into two steps: (1) Choose the seller's price points $k_1 > k_2 > \dots > k_M$ for non-garbage signals; (2) Design a signaling

scheme X maximizing $\widetilde{W}(X)$ such that the seller's best response to each signal σ is in fact k_σ .

Given the chosen price point indices k_1, \dots, k_M , the goal of finding the welfare-maximizing signaling scheme is characterized by the following linear program.

$$\begin{aligned}
& \text{Maximize} && \sum_{\sigma=1}^M \sum_{i=k_\sigma}^n v_i x_{i,\sigma} \\
& \text{subject to} && \sum_{\sigma=1}^M x_{i,\sigma} \leq p_i && \text{for all } i \quad (\text{probability}) \\
& && v_{k_\sigma} \cdot \sum_{i \geq k_\sigma} x_{i,\sigma} \geq v_k \cdot \sum_{i \geq k} x_{i,\sigma} && \text{for all } \sigma, k \quad (\text{revenue}) \\
& && x_{i,\sigma} \geq 0 && \text{for all } i, \sigma.
\end{aligned} \tag{1}$$

The probability constraints capture that we do indeed have a valid signaling scheme (with the garbage signal being assigned all residual probabilities $p_i - \sum_{\sigma=1}^M x_{i,\sigma}$). The revenue constraints capture that it is a best response for the seller to set the price point k_σ when receiving the signal σ .

Notice that for step (2), the LP (1) actually achieves the optimal solution for given price points. The approximation is necessary because for step (1), choosing the optimal price points appears more difficult.

2.4 Bayesian Persuasion

Bayesian Persuasion [18] is a natural and strong generalization of signaling in bilateral trade. The game involves a sender and a receiver, and is characterized by the following: (1) a distribution over states of nature $\omega \in \Omega$, (2) a set A of actions the receiver can choose from, and (3) utility functions $u_S(\omega, a)$, $u_R(\omega, a)$ mapping the state of nature and action chosen by the receiver to the utilities obtained by the two players.

To illustrate these abstract concepts, for the pricing game, the state of nature was the buyer's valuation, and the receiver's actions were prices P . In the present work, we assume that the distribution over states of nature is given explicitly by the vector \mathbf{p} of probabilities p_ω .

As with the pricing game, the sender can choose a signaling scheme $X = (x_{\omega,\sigma})_{\omega \in \Omega, \sigma \in \Sigma}$ with M signals. Upon receiving the signal σ , the receiver will choose an action $a(\sigma) \in A$ maximizing $\mathbb{E}_{\omega \sim \mu_\sigma} [u_R(\omega, a)]$, breaking ties in favor of the sender; here, μ_σ denotes the posterior distribution over states of nature conditioned on receiving signal σ .

The sender's utility from sending the signal σ is $u(\sigma) = \sum_\omega x_{\omega,\sigma} \cdot u_S(\omega, a(\sigma))$, and he would like to maximize his overall expected utility $U(X) = \sum_\sigma u(\sigma)$. As with the special case of social welfare, we will mostly focus on the sanitized sender utility $\widetilde{U}(X) = \sum_{\sigma \neq \perp} u(\sigma)$ of signaling schemes with a dedicated garbage signal \perp ; by the same argument as for social welfare, the sanitized sender utility is within at least a factor $\frac{M-1}{M}$ of the sender utility.

3 Welfare with Limited Communication

In this section, we provide a proof of Theorem 1.1. We then discuss some of its implications, including Corollary 1.2 and a QPTAS for the problem of maximizing social welfare subject to limited communication.

We begin by showing a special case of the theorem when each v_i is a power of 2. We then reduce the general case to this special case at a loss of a constant factor.

Lemma 3.1 *If each v_i is a power of 2, then the ratio of the full-information welfare to the maximum served social welfare of a single segment of the market is at most $O(\log n)$.*

PROOF. The lemma is trivial for $n = 1$, so assume that $n \geq 2$. Let $\text{SW} = \sum_i p_i v_i$ be the full-information welfare. First, we remove (i.e., exclude from the segment we construct) all types i with $p_i v_i < \frac{\text{SW}}{n^2}$. Because there are at most n such types, this can decrease the full-information welfare by at most a factor of $1 - 1/n$.

Now, we group all (remaining) types i into $O(\log n)$ bins according to $p_i v_i$. Specifically, bin $B_j, j \geq 0$ contains all types i with $p_i v_i \in (\frac{\text{SW}}{2^{j+1}}, \frac{\text{SW}}{2^j}]$. Since $p_i v_i \geq \frac{\text{SW}}{n^2}$, there are at most $2 \log n$ bins. Thus, there is at least one bin B_j such that $\sum_{i \in B_j} p_i v_i \geq \frac{n-1}{n} \cdot \frac{1}{2 \log n} \cdot \text{SW}$. Fix such a j for the rest of the proof, and define $u = \frac{\text{SW}}{2^{j+1}}$.

Let $i^* = \min B_j$ be the type in B_j corresponding to the smallest value. We define a market segment \mathbf{q} as follows: Let $q_{i^*} = \frac{u}{v_{i^*}}$, let $q_i = \frac{u}{2v_i}$ for all $i \in B_j$ with $i \neq i^*$, and let $q_i = 0$ for $i \notin B_j$.

By definition of the bins, for all $i \in B_j$ we have $u < p_i v_i \leq 2u$, and therefore $p_i/4 \leq q_i \leq p_i$. In particular, $\mathbf{0} \leq \mathbf{q} \leq \mathbf{p}$, and therefore \mathbf{q} describes a valid segment of \mathbf{p} .

We next show that, when facing the market segment \mathbf{q} , the seller will choose the price point i^* . The seller's revenue from choosing i^* is at least $q_{i^*} v_{i^*} = u$. On the other hand, for any $i > i^*$, the seller obtains revenue at most $v_i \cdot \sum_{k \geq i, k \in B_j} \frac{u}{2v_k} \leq v_i \cdot \frac{u}{2v_i} \cdot \sum_{k=0}^{\infty} \frac{1}{2^k} = u$; here, we used that the valuations are powers of 2. Therefore, v_{i^*} is the revenue-maximizing price for \mathbf{q} . With the seller choosing v_{i^*} as the price, the served social welfare of \mathbf{q} is

$$\sum_{i \in B_j} q_i \cdot v_i \geq \frac{1}{4} \sum_{i \in B_j} p_i \cdot v_i \geq \frac{(n-1)}{8n \log n} \cdot \text{SW} \geq \frac{1}{16n \log n} \cdot \text{SW}.$$

Thus, \mathbf{q} is a segment with served social welfare at least $\frac{1}{O(\log n)} \cdot \text{SW}$. ■

We utilize Lemma 3.1 to prove the upper-bound portion of Theorem 1.1. The idea is to round down all valuations to the nearest power of 2 (losing at most a factor of 2 in the welfare), then utilize Lemma 3.1 to construct a segment for the new valuation distribution achieving at least a factor $\Omega(1/\log n)$ of the full-information welfare, and finally reconstruct a segment/signal for the original distribution. For the matching lower bound, we use an “equal welfare distribution.”

PROOF OF THEOREM 1.1. Let $v'_j, 1 \leq j \leq m$, be the valuations of the original distribution, with associated probabilities p'_j . The new valuations are $v_i = 2^i$, with $p_i = \sum_{2^i \leq v'_j < 2^{i+1}} p'_j$. We allow the index i to be negative for notational convenience. Let n denote the support size of \mathbf{p} , and note that $n \leq m$. Because valuations were reduced by at most a factor of 2, the full-information welfare of the new distribution is at least half that of the original one, i.e., $\sum_i p_i v_i \geq \frac{1}{2} \sum_j p'_j v'_j$.

Let \mathbf{q} be a single segment with served social welfare at least a $1/O(\log n)$ fraction of the full-information welfare of \mathbf{p} . By Lemma 3.1, such a \mathbf{q} exists. Let i^* be the price point chosen by the seller under \mathbf{q} , and R the seller's revenue. By optimality of i^* for the seller, we get that

$$R = v_{i^*} \cdot (q_{i^*} + \sum_{k=i^*+1}^n q_k) \geq v_{i^*+1} \cdot \sum_{k=i^*+1}^n q_k = 2v_{i^*} \cdot \sum_{k=i^*+1}^n q_k. \quad (2)$$

Hence, $q_{i^*} \geq \sum_{k=i^*+1}^n q_k$, meaning that at least half of the probability mass, and thus half of the seller's revenue R from \mathbf{q} , comes from type i^* . Now define a segment \mathbf{q}' of the original

type distribution \mathbf{p}' as follows. For type i^* , take a total of probability mass q_{i^*} from types j with $2^{i^*} \leq v'_j < 2^{i^*+1}$. For all types $i > i^*$, take a total of probability mass $q_i/8$ from types j with $2^i \leq v'_j < 2^{i+1}$, and for types j with $v'_j < 2^{i^*}$, set $q'_j = 0$. Observe that the full-information welfare of \mathbf{q}' is at least a $1/8$ fraction of the served social welfare of \mathbf{q} . It remains to show that a constant fraction of that social welfare is above the seller's revenue-maximizing offer price for \mathbf{q}' , which we do next.

First, note that the price 2^{i^*} gives the seller a revenue of at least $R/2$, even just from all buyers of types j with $2^{i^*} \leq v'_j < 2^{i^*+1}$. Next, consider a price of $v'_j \geq 2^{i^*+1}$, say $2^i \leq v'_j \leq 2^{i+1}$ with $i \geq i^* + 1$. The revenue of such a price is at most

$$v'_j \cdot \sum_{k \geq j} q'_k \leq 2^{i+1} \cdot \sum_{k \geq i} q_k/8 = \frac{1}{4} v_i \cdot \sum_{k \geq i} q_k \leq R/4. \quad (3)$$

Hence, no price $v'_j \geq 2^{i^*+1}$ can be revenue-maximizing for \mathbf{q}' , and all types in the segment \mathbf{q}' with value at least 2^{i^*+1} are served. It remains to show that a constant factor of the remaining social welfare — associated with values between 2^{i^*} and 2^{i^*+1} — is also served.

Because the price 2^{i^*} dominates all prices $v'_j \geq 2^{i^*+1}$, the seller will choose some price j with $2^{i^*} \leq v'_j < 2^{i^*+1}$, and the chosen price must give the seller revenue at least $R/2$. The calculation in (3) implies that the revenue extracted from types j with $v'_j \geq 2^{i^*+1}$ is at most $R/4$. Hence, at least a revenue (and thus also social welfare) of $R/4$ must come from types j with $2^{i^*} \leq v'_j < 2^{i^*+1}$. By construction, the full-information welfare associated with those types is at most $2^{i^*+1} q_{i^*} = 2v_{i^*} q_{i^*}$, which is at most $2R$ by (2). Consequently, at least a $1/8$ fraction of the total social welfare associated with those types is served, as needed.

In summary, \mathbf{q}' serves a constant fraction of the full-information welfare of \mathbf{q} , which in turn is a $1/O(\log n) \geq 1/O(\log m)$ fraction of the full-information welfare of the distribution \mathbf{p}' . Hence, for any distribution supported on m types, a single segment is enough to serve a $1/O(\log m)$ fraction of the social welfare.

To show that this bound is tight, we construct an “equal-welfare distribution” with the property that no single segment extracts more than an $O(1/\log n)$ fraction of the full-information social welfare. Let $v_i = 2^i$ for $0 \leq i \leq n$, and $p_i = \frac{1}{2^i \cdot (n-i)}$ for $0 \leq i < n$, with $p_n = \frac{1}{2^n - 1} = p_{n-1}$. Notice that these “probabilities” do not sum to 1; we omit the normalization constants for legibility, since they will cancel out in the subsequent calculations.

The full-information welfare of \mathbf{p} is $\sum_{i=0}^n p_i v_i = 2 + \sum_{i=1}^n \frac{1}{i} = \Theta(\log n)$. Now consider any segment \mathbf{q} of \mathbf{p} , and let $v_k = 2^k$ be the seller's revenue-maximizing price for \mathbf{q} . If $k = n$, the served social welfare is $v_n q_n \leq v_n p_n = 2 = O(1)$, as needed. Assume now that $k < n$. For each i with $k < i \leq n$, the optimality of the price v_k implies that $v_i q_{[i,n]} \leq v_k q_{[k,n]}$, or equivalently $q_{[i,n]} \leq \frac{v_k}{v_i} q_{[k,n]}$. By choosing $i = k+1$, we get that $q_{[k+1,n]} \leq \frac{v_k}{v_{k+1}} q_{[k,n]} = \frac{q_{[k,n]}}{2}$, and consequently $q_{[k,n]} \leq 2q_k$. We can now bound the served social welfare of \mathbf{q} by a constant:

$$\begin{aligned} \sum_{i=k}^n v_i q_i &= v_k q_{[k,n]} + \sum_{i=k+1}^n (v_i - v_{i-1}) q_{[i,n]} \leq 2v_k q_k + 2 \sum_{i=k+1}^n (v_i - v_{i-1}) \frac{v_k}{v_i} q_k \\ &= 2v_k q_k \left(1 + \sum_{i=k+1}^n \frac{v_i - v_{i-1}}{v_i} \right) = v_k q_k (n + 2 - k). \end{aligned}$$

Because $q_k \leq p_k$, we can bound $v_k q_k(n+2-k) \leq v_k p_k(n+2-k) = \frac{n+2-k}{n-k} \leq 3$.

In summary, any single segment can obtain at most constant welfare, and thus at most an $O(1/\log n)$ fraction of the total social welfare for this instance. ■

To derive Corollary 1.2, simply pick segments greedily, always choosing the next segment to have largest possible served social welfare. By Theorem 1.1, each subsequent segment obtains at least a $\frac{1}{c \log n}$ fraction of the residual welfare at that point, meaning that after adding $c \log n \log(1/\epsilon)$ signals, the fraction of the total welfare obtained by the signaling scheme is at least $1 - (1 - \frac{1}{c \log n})^{c \log n \log(1/\epsilon)} \geq 1 - \epsilon$.

Remark 3.2 One can obtain a positive result very directly when the ratio $\rho = \frac{\max_i v_i}{\min_i v_i}$ is bounded. Group all buyers according to their values into $\log \rho$ bins: for $u = \min_i v_i$, buyers with $2^j u \leq v_i < 2^{j+1} u$ are put into bin j . By considering each bin as a segment, we can see that the revenue (also, the served social welfare) of each segment is at least half of the its full-information welfare. Choosing the best M bins leads to a bound of $\Omega(\min(1, M/\log \rho))$. However, our result is of more interest when the ratio between the largest and smallest valuations can be very large.

To get a quasi-polynomial time approximation scheme (QPTAS) for the problem of finding a welfare-maximizing signaling scheme with at most M signals, consider a desired approximation parameter ϵ . We distinguish two cases. If $M \leq c \log n \log(1/\epsilon)$ (again, c is the constant in $O(\log n)$ in Theorem 1.1), enumerate all possible choices of price points, and find the truly optimal one (i.e., there is no approximation factor lost in this case). Notice that there are at most $\binom{n}{M} = O(n^M)$ such combinations to consider, and for each of them, either the LP (1) or the greedy algorithm (Theorem 4.3) allows us to evaluate the maximum welfare attainable with that set of prices. Thus, the running time is quasi-polynomial for fixed ϵ .

When $M > c \log n \log(1/\epsilon)$, by Corollary 1.2, choosing the M price points greedily obtains a $(1 - \epsilon)$ fraction of the full-information welfare. Thus, it certainly obtains the same fraction of the maximum that could be achieved with M signals.

4 Sanitized Welfare Maximization with a Greedy Algorithm

In this section, we show that, given the set of price points, an optimal solution to the LP (1) (i.e., the problem of maximizing sanitized welfare) can be computed by a greedy algorithm constructing the signals one by one. Besides the faster running time, the main value of the greedy algorithm is as an analysis tool for the problem of choosing the (near-)optimal price points; the analysis for that problem is carried out in Section 5.

We begin with an algorithm for constructing just one signal σ with a given price point $k = k_\sigma$. Let \mathbf{p}' be a vector of residual probabilities for buyer types, i.e., the probability of each type that has not been allocated to any signals previously. Hence, $\mathbf{0} \leq \mathbf{p}' \leq \mathbf{p}$. The goal is to construct the probabilities \mathbf{y} for a single signal with price point k ; that is, we require that $0 \leq y_i \leq p'_i$ for all i , and the revenue constraint $v_k \cdot \sum_{j \geq k} y_j \geq v_i \cdot \sum_{j \geq i} y_j$ must be satisfied for all i . The revenue constraint can be written as

$$y_{[i,n]} \leq \frac{v_k}{v_i} \cdot y_{[k,n]} \quad \text{for all } i. \quad (4)$$

In an optimal (single) signal, for each $i \geq k$, either Inequality (4) must be tight, or $y_i = p'_i$; otherwise, y_i could be increased, raising social welfare. For the same reason, $y_k = p'_k$. Because no

buyer of type $i < k$ will ever purchase at price v_k , such buyers contribute 0 to revenue and (social or buyer) welfare. Hence, without loss of generality, the optimum signal has $y_i = 0$ for all $i < k$.

These observations suggest the following algorithm: Gradually raise y_k from 0 to p'_k . Simultaneously raise the y_i for $i > k$ in such a way that at all times, for each i , either $y_i = p'_i$ or Inequality (4) is tight. This is accomplished by raising the tail sums $y_{[i,n]}$ at a rate of $\frac{v_k}{v_i}$ while $y_{[k,n]}$ is raised at rate 1. Solving for the necessary rates of increase of individual y_i gives rise to the algorithm CONSTRUCT-ONE-SIGNAL.

Algorithm 1 Construct-One-Signal (k, v, \mathbf{p}')

```

 $\mathbf{y} \leftarrow \mathbf{0}$ 
 $I \leftarrow \{i \geq k \mid p'_i > 0\}$  {Throughout,  $I$  contains all types for which  $y_i$  can still be raised.}
while  $k \in I$  do
   $m \leftarrow \max_{i \in I} i$ 
   $\rho_m \leftarrow \frac{1}{v_m}$  {Rate of increase that keeps Inequality (4) tight for  $m$ .}
  for all  $i \in I \setminus \{m\}$  do
     $j \leftarrow \min\{i' > i \mid i' \in I\}$ 
     $\rho_i \leftarrow \frac{1}{v_i} - \frac{1}{v_j}$  {Rate of increase that keeps Inequality (4) tight for  $i$ .}
     $i^* \leftarrow \arg \min_{i \in I} (p'_i - y_i) / \rho_i$  {Index for which  $p'_{i^*} = y_{i^*}$  will become tight first.}
     $\alpha \leftarrow (p'_{i^*} - y_{i^*}) / \rho_{i^*}$  {Amount of increase in  $y_k$  until  $y_{i^*} = p'_{i^*}$ .}
    for all  $i \in I$  do
       $y_i \leftarrow y_i + \alpha \rho_i$ 
     $I \leftarrow \{i \geq k \mid y_i < p'_i\}$  {Update the set of indices that can still be raised.}
return  $\mathbf{y}$ 

```

Our first lemma simply restates the revenue constraint, and captures the fact that if $y_i < p'_i$ at any time during the algorithm, then the revenue constraint (4) must be tight for i . The proof is straightforward by induction on iterations.

Lemma 4.1 *For all indices i and every step of the algorithm, $y_{[i,n]} \leq \frac{v_k \cdot y_{[k,i]}}{v_i - v_k}$.*

Furthermore, if $i \in I$ at some point of the algorithm (including at termination), then at that point in time, $v_i \cdot y_{[i,n]} = v_k \cdot y_{[k,n]}$. The latter condition is equivalent to saying that $y_{[i,n]} = \frac{v_k y_{[k,i]}}{v_i - v_k}$.

The key property of the algorithm CONSTRUCT-ONE-SIGNAL is that it maximizes all tail sums $y_{[i,n]}$:

Lemma 4.2 *Let $\mathbf{y}' \leq \mathbf{p}'$ be any signal. If k is the chosen price by the seller under \mathbf{y}' , i.e., $v_k \cdot y'_{[k,n]} \geq v_i \cdot y'_{[i,n]}$ for all i , then $y_{[i,n]} \geq y'_{[i,n]}$ for all $i \geq k$.*

PROOF. For contradiction, assume that $y_{[i,n]} < y'_{[i,n]}$ for some i , and fix the minimum such i . Let $j \geq i$ be the minimum index with $y_j < p'_j$; note that such an index must exist, as otherwise, $y_{[i,n]} = \sum_{j \geq i} y_j = \sum_{j \geq i} p'_j \geq \sum_{j \geq i} y'_j = y'_{[i,n]}$. As a result, the index $j \in I$ at the termination of the algorithm, implying by Lemma 4.1 that $v_k \cdot y_{[k,n]} = v_j \cdot y_{[j,n]}$. Now, we obtain the following contradiction:

$$y_{[i,n]} = y_{[j,n]} + \sum_{i'=i}^{j-1} y_{i'} = \frac{v_k \cdot y_{[k,n]}}{v_j} + \sum_{i'=i}^{j-1} p'_{i'} \geq y'_{[j,n]} + \sum_{i'=i}^{j-1} y'_{i'} = y'_{[i,n]}. \blacksquare$$

To construct a complete signaling scheme, we invoke the algorithm CONSTRUCT-ONE-SIGNAL repeatedly, constructing the signals one at a time. The important part here is that the signals must be constructed in decreasing order of the target price. The intuitive reason is that the inclusion of high-value buyers in a signal makes a higher price more attractive to the seller, thus posing additional constraints on the required probability mass of lower-valued buyers that must be included. Thus, it is always better to include as many high-valued buyers in the high-priced signals as possible, and this is accomplished by constructing those signals first. Hence, we assume that the signals are sorted in descending order of their prices.

Algorithm 2 Construct-Signaling-Scheme ($v, p, S = \{k_1 > k_2 > \dots > k_M\}$)

```

 $p^{(1)} \leftarrow p$ 
for  $\sigma \leftarrow 1$  to  $M$  do
     $x_\sigma \leftarrow \text{CONSTRUCT-ONE-SIGNAL}(p^{(\sigma)}, v, k_\sigma)$   $\{x_\sigma$  is column  $\sigma$  of the signaling scheme  $X$ 's matrix. $\}$ 
     $p^{(\sigma+1)} \leftarrow p^{(\sigma)} - x_\sigma$ 
return  $X$ 

```

Theorem 4.3 *The algorithm CONSTRUCT-SIGNALING-SCHEME solves the linear program (1) optimally.*

The proof of this theorem proceeds by showing that an optimal solution X^* for the linear program (1) can be (gradually) transformed into the solution X constructed by the algorithm CONSTRUCT-SIGNALING-SCHEME without decreasing its solution quality. It is technically fairly involved, and given in Appendix A.

5 Submodularity of Sanitized Welfare

We prove that the sanitized welfare objective function $\widetilde{W}(S)$ is a submodular function of the set S of chosen price points. (There is also a garbage signal $\perp \notin S$.) Let $S = \{k_1 > k_2 > \dots > k_m\}$ be k price points. $\widetilde{W}(S)$ is defined as the optimum solution to the LP (1) with the given price points. We show the following:

Theorem 5.1 *If $S \subseteq T$ and $k \notin T$, then $\widetilde{W}(T \cup \{k\}) - \widetilde{W}(T) \leq \widetilde{W}(S \cup \{k\}) - \widetilde{W}(S)$.*

Theorem 4.3 states that the greedy algorithm solves the problem for $S \cup \{k\}$ optimally, but the effects of adding k are subtle. It is fairly easy to analyze what happens in the iteration when k itself is added: that the welfare increase is larger for S than for T is easily seen by a simple monotonicity argument, captured by Lemma 5.3. However, the addition of k has “downstream” effects. The construction of subsequent signals with price points $k_\sigma < k$ will now face different residual probabilities, and the resulting reductions in those signals need to be carefully balanced against the gains from the signal with price point k .

Part of the complexity arises from the rather complex construction of the signal for price point k itself. It is captured by the algorithm CONSTRUCT-ONE-SIGNAL, which itself runs through iterations in which different sets I of indices have their probabilities increased. In order to eliminate this source of complexity, we will think of adding the signal with price point k “gradually.”

Specifically, we consider the execution of CONSTRUCT-SIGNALING-SCHEME in which the execution of CONSTRUCT-ONE-SIGNAL for the signal with price point k may be terminated prematurely. An upper bound B on the tail probability is specified, and CONSTRUCT-ONE-SIGNAL is stopped when $\sum_{i \geq k} y_i = B$. After signal k is constructed in this modified way, subsequent signals will be constructed normally by CONSTRUCT-SIGNALING-SCHEME.

A modification of the proof of Theorem 4.3 shows that this modified algorithm optimally solves the LP (1) with the added constraint that $\sum_{i \geq k} x_{i, \sigma^k} \leq B$, where σ^k denotes the signal whose price point is k .

We write $\widetilde{W}_{(k,B)}(S)$ for the sanitized welfare achieved by the optimum solution with a set of signal price points $S \cup \{k\}$, and the constraint that the probability mass for signal k is at most B . Our main lemma is:

Lemma 5.2 *If $S \subseteq T$, then for any k, B, ϵ : $\widetilde{W}_{(k,B+\epsilon)}(T) - \widetilde{W}_{(k,B)}(T) \leq \widetilde{W}_{(k,B+\epsilon)}(S) - \widetilde{W}_{(k,B)}(S)$.*

Lemma 5.2 implies submodularity quite directly, as follows.

PROOF OF THEOREM 5.1. Let X and \widehat{X} be the optimal signaling schemes with price point sets $S \cup \{k\}$ and $T \cup \{k\}$, respectively. By Lemma 5.3, when constructing σ^k and ω^k , the residual probability for σ^k is more than that of ω^k ; therefore, $\widehat{x}_{[k,n],\omega^k} \leq x_{[k,n],\sigma^k}$ by Lemma 4.2.

Consider gradually increasing B from 0 to $\widehat{x}_{[k,n],\omega^k}$ in increments of (varying) ϵ , as outlined above. Subsequently, continue increasing B for X only. By adding up the inequality from Lemma 5.2 for each such step, and noting that the subsequent increases of B for X can only further increase the welfare of X , we obtain that $\widetilde{W}(S \cup \{k\}) - \widetilde{W}(S) \geq \widetilde{W}(T \cup \{k\}) - \widetilde{W}(T)$. ■

Because the objective function is submodular (and monotone by Lemma 5.3), the greedy algorithm is known [25] to give a $(1 - 1/e)$ -approximation for the problem of maximizing $\widetilde{W}(S)$. Proposition 2.1 then implies that the same greedy algorithm gives a $\frac{M-1}{M} \cdot (1 - 1/e)$ approximation for the objective of maximizing the social welfare $W(S)$, proving Theorem 1.3.

The proof of Lemma 5.2 is technically quite involved, and given in Appendix B. The idea is to first prove it for sufficiently small ϵ , which allows us to couple the executions tightly.

In particular, by comparing the solutions to the linear program (1), we can ensure that any constraint that becomes tight in the solution for set T with bound $B + \epsilon$, but is not tight with bound B (and similarly for S) would not have become tight for any $\epsilon' < \epsilon$. This will localize the changes, and maintain the revenue indifference for the seller. By summing over all such iterations (there will only be finitely many, because ϵ is chosen so that at least one more constraint becomes tight), we eventually prove the lemma for all ϵ .

In the analysis, we are interested in four different signaling schemes, constructed by CONSTRUCT-SIGNALING-SCHEME when run with different sets of price points and upper bounds B . We will assume here that $k \in S \subseteq T$. Specifically we define:

- $X = (x_{i,\sigma})_{\sigma,i}$: the probability mass of type i assigned to signal σ when the algorithm is run with price point set S and an upper bound of B .
- $X^+ = (x_{i,\sigma}^+)_{\sigma,i}$: probability mass for price point set S and an upper bound of $B + \epsilon$.
- $\widehat{X} = (\widehat{x}_{i,\sigma})_{\sigma,i}$: probability mass for price point set T and an upper bound of B .

- $\hat{X}^+ = (\hat{x}_{i,\sigma}^+)_{\sigma,i}$: probability mass for price point set T and an upper bound of $B + \epsilon$.

A first step of the proof, which is both illustrative of the types of arguments made repeatedly and implies monotonicity of the sanitized welfare $\bar{W}(S)$ is the following lemma. It shows that if more probability mass can be allocated for one signal, then the effect can never be a decrease in the total allocated probability mass for any type of buyer.

Lemma 5.3 (Monotonicity) *Let X, X^+ be optimal signaling schemes for price point set S , with signal set Q . Then, for any $k \in S, B, \epsilon$, and any buyer type i , we have that $x_{i,Q} \leq x_{i,Q}^+$.*

As a corollary, for any sets of price points $S \subseteq T$, and with Q and \hat{Q} as the respective sets of signals, for any B and any buyer type i , we have $x_{i,Q} \leq \hat{x}_{i,\hat{Q}}$.

PROOF. Let σ^k be the signal with price point k . Let $Q' \subseteq Q$ be an initial segment of Q , i.e., $\sigma' < \sigma$ for every $\sigma' \in Q', \sigma \in Q \setminus Q'$. We prove by induction on $|Q'|$ that $x_{i,Q'} \leq x_{i,Q'}^+$. The base case $Q' = \emptyset$ is trivial.

For the induction step, let σ' be the largest signal in Q' (i.e., with smallest price point k'), and distinguish three cases. If $\sigma' < \sigma^k$, i.e., before σ^k is constructed, the execution of CONSTRUCT-SIGNALING-SCHEME is the same for the bounds B and $B + \epsilon$, so $x_{i,\sigma'} = x_{i,\sigma'}^+$, and the induction follows directly.

When $\sigma' = \sigma^k$, x_{i,σ^k} is the result of CONSTRUCT-ONE-SIGNAL with an upper bound of B , and x_{i,σ^k}^+ is that with upper bound of $B + \epsilon$. Until the tail sum reaches B , the execution is the same, and subsequently, values can only be raised further for the execution with the bound $B + \epsilon$. Thus, $x_{i,\sigma^k} \leq x_{i,\sigma^k}^+$, and again, the induction step follows.

For $\sigma' > \sigma^k$, assume for contradiction that there exists an i such that $x_{i,Q'} > x_{i,Q'}^+$; fix the smallest such i . Since $p_i \geq x_{i,Q'} > x_{i,Q'}^+$, we get that the index i is in I for the execution with upper bound $B + \epsilon$ all the way until σ' is constructed, implying by Lemma 4.1 (and the revenue constraint for the upper bound of B) that for all $\sigma \leq \sigma'$,

$$x_{[i,n],\sigma} \leq \frac{v_{k_\sigma}}{v_i - v_{k_\sigma}} \cdot x_{[k_\sigma,i],\sigma}, \quad x_{[i,n],\sigma}^+ = \frac{v_{k_\sigma}}{v_i - v_{k_\sigma}} \cdot x_{[k_\sigma,i],\sigma}^+.$$

By summing over all $\sigma \in Q'$, we also obtain that

$$x_{[i,n],Q'} \leq \sum_{\sigma \in Q'} \frac{v_{k_\sigma}}{v_i - v_{k_\sigma}} \cdot x_{[k_\sigma,i],\sigma}, \quad x_{[i,n],Q'}^+ = \sum_{\sigma \in Q'} \frac{v_{k_\sigma}}{v_i - v_{k_\sigma}} \cdot x_{[k_\sigma,i],\sigma}^+.$$

By minimality of i , we have that $x_{i',Q'} \leq x_{i',Q'}^+$ for all $i' < i$; summing these inequalities gives that $x_{[k_\sigma,i],Q'} \leq x_{[k_\sigma,i],Q'}^+$. Similarly, for any initial segment $Q'' \subsetneq Q'$, the strong induction hypothesis implies that $x_{i',Q''} \leq x_{i',Q''}^+$ for all i' (in particular, $i' < i$); summing those inequalities, and combining with the one just derived proves that $x_{[k_\sigma,i],Q''} \leq x_{[k_\sigma,i],Q''}^+$ for all initial segments $Q'' \subseteq Q'$ (including $Q'' = Q'$).

Because $\frac{v_{k_\sigma}}{v_i - v_{k_\sigma}}$ is monotone non-increasing in σ (recall that k_σ is decreasing), we can apply Lemma B.5 in the middle step of the following derivation:

$$x_{[i,n],Q'}^+ = \sum_{\sigma \in Q'} \frac{v_{k_\sigma}}{v_i - v_{k_\sigma}} \cdot x_{[k_\sigma,i],\sigma}^+ \stackrel{\text{Lemma B.5}}{\geq} \sum_{\sigma \in Q'} \frac{v_{k_\sigma}}{v_i - v_{k_\sigma}} \cdot x_{[k_\sigma,i],\sigma} \geq x_{[i,n],Q'}. \quad (5)$$

Because $x_{i,Q'} > x_{i,Q'}^+$, but $x_{[i,n],Q'} \leq x_{[i,n],Q'}^+$, there must be some $j > i$ such that $x_{j,Q'} < x_{j,Q'}^+$; fix a minimal such j . This time, since $p_j \geq x_{j,Q'}^+ > x_{j,Q'}$, we can apply Lemma 4.1 (and the revenue constraint for the process with bound $B + \epsilon$) to obtain that for all $\sigma \leq \sigma'$,

$$v_j \cdot x_{[j,n],\sigma} = v_{k_\sigma} \cdot x_{[k_\sigma,n],\sigma} \geq v_i \cdot x_{[i,n],\sigma}, \quad v_j \cdot x_{[j,n],\sigma}^+ \leq v_{k_\sigma} \cdot x_{[k_\sigma,n],\sigma}^+ = v_i \cdot x_{[i,n],\sigma}^+.$$

Solving for $x_{[j,n],\sigma}$ and $x_{[j,n],\sigma}^+$, we obtain that $x_{[j,n],\sigma} \geq \frac{v_j}{v_j - v_i} \cdot x_{[i,j],\sigma}$ and $x_{[j,n],\sigma}^+ \leq \frac{v_j}{v_j - v_i} \cdot x_{[i,j-1],\sigma}^+$. Summing over all $\sigma \in Q'$ now gives us that

$$x_{[j,n],Q'} \geq \frac{v_j}{v_j - v_i} \cdot x_{[i,j],Q'}, \quad x_{[j,n],Q'}^+ \leq \frac{v_j}{v_j - v_i} \cdot x_{[i,j],Q'}^+. \quad (6)$$

By the definition of i and j , we have that $x_{[i,j],Q'} > x_{[i,j],Q'}^+$; substituting this inequality into (6) and canceling common terms implies that $x_{[j,n],Q'} > x_{[j,n],Q'}^+$. Now, we derive a contradiction as follows: by Inequality (5), we have

$$x_{[j,n],Q'} = x_{[i,n],Q'} - x_{[i,j],Q'} \stackrel{(5)}{<} x_{[i,n],Q'}^+ - x_{[i,j],Q'}^+ = x_{[j,n],Q'}^+ < x_{[j,n],Q'}. \blacksquare$$

6 Revenue in Bilateral trade

In this section, we prove Theorem 1.4, giving a straightforward dynamic program to compute a signaling scheme maximizing the seller's revenue. Before doing so, we exhibit an equal-revenue distribution for which any signaling scheme with M signals only recovers an $O(M/n)$ fraction of the full-information welfare. Notice again the contrast to the case of welfare maximization, where even one segment is enough to attain an $\Omega(1/\log n)$ fraction of the full-information welfare.

We define an equal-revenue distribution as follows. Let the valuations be $v_i = 2^i$ for $0 \leq i \leq n$, and the probabilities $p_i = \frac{1}{2^{i+1}}$ for $0 \leq i < n$, and $p_n = \frac{1}{2^n}$. The full-information welfare is $\sum_{i=0}^n p_i v_i = \frac{n}{2} + 1$. However, for every segment $\mathbf{q} \leq \mathbf{p}$, no matter what price v_i the seller chooses, the revenue cannot be more than $v_i \cdot \sum_{j \geq i} p_j = 1$. Therefore, in the worst case, with M signals, the seller can at best get an $O(M/n)$ fraction of the maximum social welfare as his revenue, whereas a fully informed seller would be able to extract the entire full-information welfare, as discussed in the introduction.

Next, we turn our attention to the problem of computing the optimum signaling scheme for the seller's revenue. The key insight enabling a dynamic program is that the seller-optimal signaling scheme partitions the buyer types into disjoint intervals, and allocates all probability mass for a given interval to one signal.

Lemma 6.1 (Interval Structure of Seller-Optimal Signaling Scheme) *W.l.o.g., the seller-optimal signaling scheme X has the following form: There are disjoint intervals I_1, I_2, \dots, I_M of buyer types such that $\bigcup_{\sigma} I_{\sigma} = \{1, \dots, n\}$, and for each signal σ , $x_{i,\sigma} = p_i$ for all $i \in I_{\sigma}$ (and $x_{i,\sigma} = 0$ for all $i \notin I_{\sigma}$).*

PROOF. Let $k_1 > k_2 > \dots > k_M$ be the price points of the signals σ under X . We will show how to transform X to the claimed form without decreasing the seller's revenue.

First, if $x_{i,\sigma} > 0$ for some $\sigma < M, i < k_\sigma$, then the buyers of type i will not buy when signal σ is sent, contributing nothing to the seller's revenue. Therefore, setting $x_{i,\sigma} = 0$ instead does not lower the seller's revenue, and increasing $x_{i,M}$ by the same amount again cannot decrease the seller's revenue. Hence, we may assume that for all signals $\sigma < M$, we have $x_{i,\sigma} > 0$ only for $i \geq k_\sigma$.

Next, if $x_{i,\sigma} > 0$, then $x_{i,\sigma} = p_i$. We distinguish two cases: if there is unallocated probability mass of type i , then $x_{i,\sigma}$ can simply be raised. If $x_{i,\sigma'} > 0$ for $\sigma' > \sigma$, we can lower $x_{i,\sigma'}$ to 0 while raising $x_{i,\sigma}$ by the same amount. Because $\sigma < \sigma' \leq M$, we have that $i \geq k_\sigma$, so the seller's revenue increases by $x_{i,\sigma'} \cdot (v_{k_\sigma} - v_{k_{\sigma'}}) \geq 0$.

So far, we have shown that the signals partition the buyer types into sets such that for each buyer type, all of its probability mass goes to its unique designated signal. It remains to show that the partitions are intervals. If not, then there would be two signals $\sigma' > \sigma$ and price points $i < i'$ such that $x_{i,\sigma} = p_i, x_{i',\sigma'} = p_{i'}$. Then, reallocating the probability mass $x_{i',\sigma'}$ to signal σ instead increases the seller's revenue by at least $x_{i',\sigma'} \cdot (v_{k_\sigma} - v_{k_{\sigma'}}) \geq 0$. ■

The dynamic program for segmentation into intervals is now standard. Let $R(i, m)$ denote the optimal revenue a seller can obtain from buyer types $\{i, i+1, \dots, n\}$ with m signals, when the lowest price is v_i . $R(i, m)$ satisfies the recurrence $R(i, 0) = 0$ and $R(i, m) = \max_{i < i' \leq n} (R(i', m-1) + v_i \cdot \sum_{j=i}^{i'-1} p_j)$. The maximum attainable revenue can be found by exhaustive search of $R(i, M)$ over all i .

7 Hardness of General Persuasion

In this section, we present the proof of Theorem 1.5 from the introduction. More accurately, the following theorem shows the hardness of maximizing sanitized sender utility within any constant.

Theorem 7.1 *Unless $P = NP$, for any constant $c > 0$, there is no polynomial-time algorithm for the following problem. Given a Bayesian persuasion game $(\Omega, \mathbf{p}, A, u_S, u_R)$ and cardinality constraint M on the number of signals, construct a signaling scheme X using at most M signals such that the sanitized sender utility $\tilde{U}(X)$ under X is at least $c \cdot \tilde{U}(X^*)$, where X^* is the signaling scheme maximizing $\tilde{U}(X)$.*

Because the sender utility and sanitized sender utility are within a factor of $\frac{M-1}{M}$ of each other, this implies the same hardness result for the sender utility, proving Theorem 1.5.

We prove Theorem 7.1 by establishing hardness for a game we call the HYPERGRAPH EDGE GUESSING GAME (HEGG). There is a hypergraph $H = (V, E)$ which is commonly known to the sender and receiver. The state of nature is a hyperedge $e^* \in E$, drawn from the uniform distribution.

The receiver has two types of actions available: trying to guess the hyperedge, or “hedging her bets” by guessing a vertex $v \in V$. If she guesses an edge e , then she gets 1 if her guess was correct ($e = e^*$), and 0 otherwise. If she guesses a vertex v , she gets $1/d_v$ (the degree of v) if v is incident on e^* , and 0 otherwise.

The sender's utility is determined by the receiver's guess. If the receiver guesses an edge, the sender gets utility 0, regardless of whether the guess is correct. If the receiver guesses a vertex v , the sender has utility $1/d_v$ (the same as the receiver) if v is incident on e^* , and 0 otherwise.

Since the sender has access to e^* , it is his goal to design a signaling scheme that narrows down the possible states of nature for the receiver enough that she can get an incident vertex, but not so much as to induce her to guess a hyperedge. This is accomplished by making the posterior

distribution conditioned on any signal uniform across edges incident on a particular vertex. Ideally, we would like this to be the case for all signals, but this may simply be impossible. However, we can achieve it for all but one signal.

Definition 7.2 *A signaling scheme X is vertex-centric if for all signals σ except at most one, there exists a node $v = v(\sigma)$ such that $x_{v,e} = x_{v,e'}$ for all hyperedges $e, e' \ni v$, and $x_{v,e} = 0$ for all hyperedges $e \not\ni v$.*

That is, in a vertex-centric signaling scheme, all but one signal induce a uniform posterior distribution over edges incident on one vertex.

Lemma 7.3 *For any signaling scheme X , there is a vertex-centric signaling scheme X' with $\tilde{U}(X') \geq \tilde{U}(X)$, and which can be constructed from X in polynomial time.*

PROOF. Consider any signaling scheme X , characterized by the probabilities $x_{e,\sigma}$ that the state of the world is e and the sender sends the signal σ . (Recall that these are not conditional probabilities.) For each signal σ , there is a unique (after tiebreaking) action that the receiver takes, either a hyperedge e or a vertex v . If the receiver chooses a hyperedge e , the sender's utility is 0; for a vertex v , it is $u(\sigma) = \frac{1}{d_v} \cdot \sum_{e \text{ incident on } v} x_{e,\sigma}$.

Let \perp be the designated garbage signal; without loss of generality (by renaming), it minimizes $u(\perp)$. First, we may assume w.l.o.g. that the receiver does not choose a hyperedge for any signal $\sigma \neq \perp$. Otherwise, since the sender's utility $u(\sigma) = 0$, we could reallocate all probability mass from σ to \perp without changing the sanitized sender utility; under the new signal (which is never sent), the receiver w.l.o.g. plays a vertex.

Consider any signal $\sigma \neq \perp$, and let v be the vertex the receiver chooses in response to σ . First, if there is any hyperedge e not incident on v with $x_{e,\sigma} > 0$, we can safely lower $x_{e,\sigma}$ to 0 (reassigning the probability mass to \perp), without changing the receiver's action (because e was not incident on v , this change cannot make v less attractive), and without affecting $u(\sigma)$.

Let $d = d_v$ be the degree of v , and e_1, e_2, \dots, e_d the hyperedges incident on v , sorted such that $x_{e_1,\sigma} \geq x_{e_2,\sigma} \geq \dots \geq x_{e_d,\sigma}$. If $x_{e_1,\sigma} > x_{e_d,\sigma}$, then the receiver's expected utility from choosing e_1 is $x_{e_1,\sigma}$, whereas her utility from choosing v is $\frac{1}{d} \sum_{i=1}^d x_{e_i,\sigma} < x_{e_1,\sigma}$. This would contradict the receiver's playing v .

Notice that the changes do not affect the utility under any signal except the garbage signal, so the sanitized sender utility stays the same. \blacksquare

Now consider the following optimization problem: find a vertex-centric signaling scheme X with a dedicated garbage signal \perp that maximizes the sanitized sender utility $\tilde{U}(X)$. By definition of a vertex-centric signaling scheme, $x_{e,v} = x_{e',v}$ for all hyperedges e, e' incident on v ; we denote this quantity by y_v . Then, the probability of sending the signal inducing the receiver to choose v is $\sum_{e \ni v} x_{e,v} = d_v y_v$, and the resulting sender utility conditioned on sending it is $1/d_v$. A vertex-centric signaling scheme is entirely determined by the $M - 1$ vertices and their associated probabilities y_v ; hence, the optimization problem can be expressed as follows.

$$\begin{aligned} & \text{Maximize} && \|y\|_1 \\ & \text{subject to} && \sum_{v \in e} y_v \leq \frac{1}{|E|} \quad \text{for all } e \in E, \\ & && \|y\|_0 \leq M - 1, \\ & && y \geq \mathbf{0}. \end{aligned}$$

The first constraint captures that the total probability of all signals sent when the state of the world is e can be at most the probability that the state of the world is e , which is $1/|E|$. Rescaling all y_v values by a factor $|E|$ and removing that constant factor from the objective gives us the following equivalent characterization.

$$\begin{aligned} & \text{Maximize} && \|\mathbf{y}\|_1 \\ & \text{subject to} && \sum_{v \in e} y_v \leq 1 \quad \text{for all } e \in E, \\ & && \|\mathbf{y}\|_0 \leq M - 1, \\ & && \mathbf{y} \geq \mathbf{0}. \end{aligned} \tag{7}$$

Notice that Program (7) would exactly be an Independent Set characterization if the y_v were restricted to be integral. The following lemma shows that the upper bound on the support of \mathbf{y} is enough to ensure that the optimal solution cannot be approximated to within any constant (when the hyperedges are large enough).

Lemma 7.4 *For any constant $r \geq 1$, unless $P = NP$, the optimum solution of Program (7) cannot be approximated to within a factor better than $1/r$.*

PROOF. We give a reduction from the gap version of INDEPENDENT SET. Given a graph $G = (V, E)$ and any constant $\epsilon > 0$, and the promise that the largest independent set of G has size either less than n^ϵ or more than $n^{1-\epsilon}$, it is NP-hard to answer “No” in the former case and “Yes” in the latter [17]. For our reduction, we specifically choose $\epsilon = \frac{1}{r+2}$.

Given G , we create a hypergraph $H = (V, E')$ on the same node set, whose hyperedges are exactly the cliques of size $r + 1$ in G , i.e., $E' = \{S \subseteq V \mid S \text{ is a clique of size } r + 1 \text{ in } G\}$. The constraint on the support size of \mathbf{y} is $M' = n^{1-\epsilon}$. Notice that the reduction is computed in time $O(n^{r+1})$, which is polynomial in n for constant r . We will show that if G has an independent set of size $n^{1-\epsilon}$, then the objective value of Program (7) is M' , whereas if G has no independent set of size n^ϵ , then the objective value is less than M'/r .

First, suppose that G has an independent set S of size M' . Consider the solution \mathbf{y} to Program (7) which sets $y_v = 1$ for all $v \in S$, and $y_v = 0$ for all others. Because S is independent in G , it contains at most one vertex from each $(r + 1)$ -clique; hence, the proposed solution is valid, and it achieves an objective value of M' .

Conversely, let \mathbf{y} be a solution to Program (7), and assume that its objective value is at least $\frac{1}{r} \cdot M'$. Let S be the set of all indices v such that $y_v > \frac{1}{r+1}$. Then, by the assumed lower bound on the objective value, $|S| \geq \frac{M'}{r+1}$.

Consider the subgraph $G[S]$ induced by S in G . By the constraint for each hyperedge, $G[S]$ contains no $(r + 1)$ -clique; otherwise, the corresponding y_v would add up to more than 1. Now, Ramsey’s Theorem implies that $G[S]$ contains an independent set of size $\Omega(n^{1/r})$, as follows. Recall that the Ramsey Number $R(r + 1, b)$ is the minimum size s such that each graph of size s contains a clique of size $r + 1$ or an independent set of size b . Because $R(r + 1, b) \leq \binom{r+b-1}{r} \in O(b^r)$ [29], and $G[S]$ is a graph of size at least $\frac{M'}{r+1}$ not containing any $(r + 1)$ -clique, it must contain an independent set of size at least $\Omega((M'/r^2)^{1/r}) = \Omega(n^{(1-\epsilon)/r}) = \omega(n^\epsilon)$. This completes the proof. ■

PROOF OF THEOREM 7.1. The proof is now straightforward. Given an instance of the INDEPENDENT SET problem, we construct the instance of the HEGG according to the proof of Lemma 7.4,

setting the allowed number of signals to $M = 1 + M'$. Feasible solutions to Program (7) exactly capture vertex-centered signaling schemes, and the objective value is the sanitized sender utility (scaled by $|E'|$). ■

8 Conclusion

Our work raises several natural questions for future work. First, while we provide a constant-factor approximation algorithm and a QPTAS for social welfare in the pricing game, we did not actually establish NP-hardness. Is there a polynomial-time algorithm for maximizing social welfare subject to limited communication? Could at least a PTAS or an FPTAS be obtained? For the more general persuasion problem with limited communication, we establish that no approximation of social welfare to within any constant is possible. Can this result be strengthened to logarithmic or polynomial hardness?

In the present submission, we are focusing on maximizing seller revenue and social welfare. Bergemann et al. [3] also consider maximizing buyer’s utility. It is not hard to see that given the price points for each signal, the buyer’s utility can be maximized by a linear program very similar to (1). However, it is not clear that the overall objective function is still submodular, or whether a similar greedy algorithm to the one from Section 4 optimally solves the corresponding LP.

Beyond revenue and social or buyer welfare, one could consider other objectives for the principal. While for full generality of the persuasion problem, our results preclude constant-factor approximation guarantees, it would be of interest to identify other natural classes in which limits on communication have mild consequences, and in which good signaling schemes with limited communication can be designed efficiently.

Acknowledgments.

This work is supported in part by NSF grant CCF-1423618. We would also like to thank Alex Eager for useful conversations and anonymous reviewers for their constructive comments and suggestions.

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A Proof of Theorem 4.3

In this section, we prove Theorem 4.3.

Theorem 4.3 *The algorithm CONSTRUCT-SIGNALING-SCHEME solves the linear program (1) optimally.*

PROOF. Let X^* be an optimal solution for the linear program (1), and X the signaling scheme constructed by the algorithm CONSTRUCT-SIGNALING-SCHEME. We will show that X^* can be (gradually) transformed into X without decreasing its solution quality, proving optimality of X .

First, we may assume without loss of generality that $x_{i,\sigma}^* = 0$ for all $i < k_\sigma$, since setting them to 0 affects neither the objective value nor the constraints. Let $\mathbf{x}_\sigma, \mathbf{x}_\sigma^*$ denote column σ of X, X^* , i.e., the vector of probabilities that constitute signal σ .

Assume that $X \neq X^*$. Let σ be the smallest index (i.e., with largest price) such that $\mathbf{x}_\sigma \neq \mathbf{x}_\sigma^*$. Let i be minimal such that $x_{i,\sigma} \neq x_{i,\sigma}^*$. For notational convenience, since we will mostly focus on

the signal σ , we write $\mathbf{y} = \mathbf{x}_\sigma$ and $\mathbf{y}^* = \mathbf{x}_\sigma^*$. Let $\mathbf{p}' = \mathbf{p} - \sum_{\sigma' < \sigma} \mathbf{x}_{\sigma'} = \mathbf{p} - \sum_{\sigma' < \sigma} \mathbf{x}_{\sigma'}^*$ be the vector of residual probabilities at the time that signal σ was greedily constructed. We now distinguish two cases:

Case 1: $y_i < y_i^*$:

By Lemma 4.2, $y_{[i,n]} \geq y_{[i,n]}^*$, and because $y_i < y_i^*$, we get that $y_{[i+1,n]} > y_{[i+1,n]}^*$. In particular, there must be an index $i' > i$ such that $y_{i'} > y_{i'}^*$; let i' be the smallest such index. Let $\delta = \min(y_{i'} - y_{i'}^*, y_i^* - y_i) > 0$.

We next show that under X^* , all signals combined must use all probability mass of type i' , i.e., $\sum_{\sigma'=1}^M x_{i',\sigma'}^* = p_{i'}$. If this were not the case, then define $\epsilon = \min(\delta, p_{i'} - \sum_{\sigma'=1}^M x_{i',\sigma'}^*) > 0$, and consider modifying X^* by updating $y_{i'}' = y_{i'}^* + \epsilon$ and $y_i' = y_i^* - \epsilon$ (and leaving $y_j' = y_j^*$ for all $j \neq i, i'$). By choice of ϵ , this new solution X' does not violate the non-negativity or total probability constraints, and we claim that (1) it satisfies the revenue constraint (4) for all j , and (2) its welfare is strictly higher than that of X^* .

To check the revenue constraints, notice first that the seller's revenue under X' for indices $j \leq i$ and $j > i'$ is unchanged, so (4) still holds for such j . It remains to consider $j \in \{i+1, \dots, i'\}$. Fix one such j . By definition of i and i' , we have that $y_{j'} \leq y_{j'}'$ for $k_\sigma \leq j' < i'$; in particular, we can infer for our j that

$$y'_{[k_\sigma, j]} = \sum_{j'=k_\sigma}^{j-1} y'_{j'} \geq \sum_{j'=k_\sigma}^{j-1} y_{j'} = y_{[k_\sigma, j]}. \quad (8)$$

Because $y_{[i,n]} \geq y_{[i,n]}^*$ and $y_i \leq y_i^* - \epsilon$, and the definition of i' , we get that for all $j \in \{i+1, \dots, i'\}$,

$$y_{[j,n]} \geq y_{[j,n]}^* + \epsilon = y'_{[j,n]}. \quad (9)$$

Combining these two inequalities with the fact that X satisfies all revenue constraints, in particular $v_j \cdot y_{[j,n]} \leq v_{k_\sigma} \cdot y_{[k_\sigma, n]}$, we get that

$$v_{k_\sigma} \cdot y'_{[k_\sigma, j]} \stackrel{(8)}{\geq} v_{k_\sigma} \cdot y_{[k_\sigma, j]} \geq (v_j - v_{k_\sigma}) \cdot y_{[j,n]} \stackrel{(9)}{\geq} (v_j - v_{k_\sigma}) \cdot y'_{[j,n]}.$$

Adding $v_{k_\sigma} \cdot y'_{[j,n]}$ to both sides now shows that X' satisfies the revenue inequality for j . However, notice that the objective value has increased by $\epsilon(v_{i'} - v_i)$, contradicting the optimality of X^* . Hence, we have shown that $\sum_{\sigma'=1}^M x_{i',\sigma'}^* = p_{i'}$. We will show that there is another signal $\sigma' > \sigma$ such that we can redistribute probability mass between signals σ, σ' without affecting either the objective or the constraints, while making X^* and X more similar.

Because $y_{i'} > y_{i'}^*$ and $\sum_{\sigma'=1}^M x_{i',\sigma'}^* = p_{i'}$, there must be a signal $\sigma' > \sigma$ such that $x_{i',\sigma'}^* > x_{i',\sigma'}$. Fix σ' to be the smallest such index, and define $\epsilon = \min(x_{i',\sigma'}^*, \delta)$. Consider the modified signaling

scheme X' with

$$\begin{aligned}
y'_i &= y_i^* - \epsilon, \\
y'_{i'} &= y_{i'}^* + \epsilon, \\
x'_{i,\sigma'} &= x_{i,\sigma'}^* + \epsilon, \\
x'_{i',\sigma'} &= x_{i',\sigma'}^* - \epsilon, \\
x'_{j,\sigma''} &= x_{j,\sigma''}^* \quad \text{for all other } j, \sigma''.
\end{aligned}$$

Because this assignment only redistributes probability mass, the probability mass constraints cannot be violated, and the social welfare stays the same. Non-negativity follows from the choice of ϵ . That \mathbf{y}' satisfies the revenue constraints follows exactly the same proof as in the previous definition of \mathbf{y}' , because the adjustment is the same. Finally, for all $j \in \{i+1, \dots, i'\}$, the tail sums of probabilities have decreased under σ' , implying that so has the revenue for those prices. This means that all revenue constraints are also satisfied for σ' .

This modification makes y_i and $y_{i'}^*$ more similar, and repeating this procedure with another signal σ' as needed, they will eventually become the same.

Case 2: $y_i > y_i^*$:

Analogous to the first case, we will begin by showing that $\sum_{\sigma'=1}^M x_{i,\sigma'}^* = p_i$. For contradiction, assume that $\sum_{\sigma'=1}^M x_{i,\sigma'}^* < p_i$, and let $\epsilon = \min(y_i - y_i^*, p_i - \sum_{\sigma'=1}^M x_{i,\sigma'}^*)$. Define an improved signaling scheme by setting $y'_i = y_i^* + \epsilon$, and $y'_j = y_j^*$ for all $j \neq i$.

The probability mass constraints are still satisfied, and nothing has changed for indices $j > i$, so those revenue constraints are still satisfied. Next, consider an index $j \in \{k_\sigma + 1, \dots, i\}$. Because i is the first index with $y_i \neq y_i^*$, we get that $y_{[k_\sigma, j]}^* = y_{[k_\sigma, j]}$.

By Lemma 4.2, $y_{[i+1, n]} \geq y_{[i+1, n]}^*$, and because $y_i \geq y_i^* + \epsilon$, we obtain that $y_{[j, n]} \geq y_{[j, n]}^* + \epsilon = y'_{[j, n]}$. Combining this and the previous inequality with the revenue constraint for \mathbf{y} at index j (namely, that $v_{k_\sigma} \cdot y_{[k_\sigma, n]} \geq v_j \cdot y_{[j, n]}$), we conclude — analogously to the previous case — that

$$v_{k_\sigma} \cdot y'_{[k_\sigma, j]} = v_{k_\sigma} \cdot y_{[k_\sigma, j]} \geq (v_j - v_{k_\sigma}) \cdot y_{[j, n]} \geq (v_j - v_{k_\sigma}) \cdot y'_{[j, n]}.$$

Adding $v_{k_\sigma} \cdot y'_{[j, n]}$ to both sides now establishes that the revenue constraint is satisfied at index j . Because the objective value strictly increased, we obtain a contradiction to the optimality of X^* . Hence, from now on, we assume that $\sum_{\sigma'=1}^M x_{i,\sigma'}^* = p_i$.

As in Case 1, there must be some signal $\sigma' > \sigma$ with $x_{i,\sigma'}^* > 0$. Our goal is again to increase y_i^* to make it closer to y_i , and do so by reassigning probability mass from signal σ' . However, in this case, doing so involves a more careful reallocation of probability mass to ensure all revenue constraints are satisfied. In fact, we will use the algorithm CONSTRUCT-ONE-SIGNAL to construct a vector \mathbf{d} describing the probability reallocation, and then set

$$\begin{aligned}
\mathbf{y}' &= \mathbf{y}^* + \mathbf{d}, \\
\mathbf{x}'_{\sigma'} &= \mathbf{x}_{\sigma'}^* - \mathbf{d}.
\end{aligned}$$

The “residual probability vector” \mathbf{r} in this case is defined as $r_i = \min(x_{i,\sigma'}^*, y_i - y_i^*)$ and $r_j = x_{j,\sigma'}^*$ for $j \neq i$. It captures the fact that we can at most reallocate all of the probability mass of $\mathbf{x}_{\sigma'}^*$,

but also must ensure that the new signal satisfies $y'_i \leq y_i$. Since the modified version of signal σ must satisfy the revenue constraints with target price v_{k_σ} , we make this price the target of the construction of \mathbf{d} , by setting $u_i = v_{k_\sigma}$, $u_j = v_j$ for $j > i$, and $u_j = 0$ for $j < i$. Then, the change vector is defined as $\mathbf{d} = \text{CONSTRUCT-ONE-SIGNAL}(\mathbf{r}, \mathbf{u}, i)$. We now want to show that the signals $\mathbf{y}', \mathbf{x}'_{\sigma'}$, defined above satisfy all constraints.

First, because probability mass only gets moved around between signals, the welfare stays the same, and the probability and non-negativity constraints cannot get violated because $\text{CONSTRUCT-ONE-SIGNAL}$ at most uses r_j units of probability in coordinate j . We therefore focus on verifying the revenue constraints.

We begin with $\mathbf{x}'_{\sigma'}$, which only saw its probabilities decrease. The seller's revenue at the target index $k_{\sigma'}$ decreased by $v_{k_{\sigma'}} \cdot d_{[i,n]}$. Consider any index $j > k_{\sigma'}$. The seller's revenue at index j decreased by $v_j \cdot d_{[j,n]}$. If $j \notin I$ at the termination of the algorithm, then $d_j = r_j$, and $x'_{j,\sigma'} = 0$, meaning that price j is no more attractive than $j+1$ or $j-1$ to the seller. Otherwise, Lemma 4.1 implies that

$$v_j \cdot d_{[j,n]} = u_j \cdot d_{[j,n]} \stackrel{\text{Lemma 4.1}}{=} u_i \cdot d_{[i,n]} = v_{k_\sigma} \cdot d_{[i,n]} \geq v_{k_{\sigma'}} \cdot d_{[i,n]}.$$

In particular, j cannot have become more attractive to the seller than $k_{\sigma'}$.

We next verify that the revenue constraints are also satisfied for the signal \mathbf{y}' . Here, the seller's revenue for price point j increased by $v_j \cdot d_{[j,n]}$. For all $j > i$, the algorithm $\text{CONSTRUCT-ONE-SIGNAL}$ ensures that $v_j \cdot d_{[j,n]} = u_j \cdot d_{[j,n]} \leq u_i \cdot d_{[i,n]} = v_{k_\sigma} \cdot d_{[i,n]}$, so the increase in revenue is no larger for price point j than for k_σ . Thus, we have obtained the revenue constraint $v_{k_\sigma} \cdot y'_{[k_\sigma,n]} \geq v_j \cdot y'_{[j,n]}$. By rewriting $y'_{[k_\sigma,n]} = y'_{[j,n]} + (y'_{[k_\sigma,n]} - y'_{[j,n]})$ and rearranging, we obtain the useful form

$$y'_{[j,n]} \leq \frac{v_{k_\sigma} \cdot y'_{[k_\sigma,j]}}{v_j - v_{k_\sigma}}. \quad (10)$$

The slightly tricky part is the indices $j \in \{k_\sigma + 1, \dots, i\}$. While the probabilities y'_j for $j < i$ do not increase, the tail probabilities $y'_{[j,n]}$ do by virtue of increases in y'_j for $j \geq i$; hence, we need to also consider these price points. Fix such a $j \in \{k_\sigma + 1, \dots, i\}$. We will first show that $y'_{[j,n]} \leq y_{[j,n]}$.

Let $i' > i$ be a smallest index with $y_{i'} < p_{i'} - \sum_{\sigma'' < \sigma} x_{i',\sigma''}$. (If no such i' exists, then let $i' = n+1$.) We will first show that $y'_{[i',n]} \leq y_{[i',n]}$. This holds trivially when $i' = n+1$. Otherwise, we apply Lemma 4.1 to the construction of \mathbf{y} , and obtain that $y_{[i',n]} = \frac{v_{k_\sigma} \cdot y_{[k_\sigma,i']}}{v_{i'} - v_{k_\sigma}}$. Now, notice that for all $j' \in \{k_\sigma, \dots, i' - 1\}$, we have that $y'_{j'} \leq y_{j'}$, for different reasons.

1. For $j' > i$, this follows because the definition of i' implies that $y_{j'} = p_{j'} - \sum_{\sigma'' < \sigma} x_{j',\sigma''}$ is as large as it can possibly be.
2. For $j' = i$, it follows because $y'_i = y_i^* + \epsilon \leq y_i$.
3. For $j' < i$, it follows because $y'_{j'} = y_{j'}^* = y_{j'}$.

This implies that $y'_{[k_\sigma,i']} \leq y_{[k_\sigma,i']}$, and hence — using Inequality (10) — that $y'_{[i',n]} \leq y_{[i',n]}$. Finally, the previous three cases show that for the fixed j ,

$$y'_{[j,n]} = y'_{[i',n]} + \sum_{j'=j}^{i'-1} y'_{j'} \leq y_{[i',n]} + \sum_{j'=j}^{i'-1} y_{j'} = y_{[j,n]}.$$

Having shown that $y'_{[j,n]} \leq y_{[j,n]}$, we next apply Lemma 4.1 to \mathbf{y} at price point j to obtain that $y_{[j,n]} \leq \frac{v_{k\sigma} \cdot y_{[k\sigma,j]}}{v_j - v_{k\sigma}}$. Combining this with the fact that $y_{[k\sigma,j]} = y'_{[k\sigma,j]}$ for $j < i$ by the third case of the above case distinction, we obtain that $y'_{[j,n]} \leq \frac{v_{k\sigma} \cdot y'_{[k\sigma,j]}}{v_j - v_{k\sigma}}$, which is an equivalent way of rewriting the revenue constraint.

Again, this modification makes y_i and y_i^* more similar, and repeating this procedure with additional signals σ' as needed, they will eventually become the same. \blacksquare

B Proof of Lemma 5.2

In this section, we provide a proof of Lemma 5.2, restated here for convenience.

Lemma 5.2 *If $S \subseteq T$, then for any k, B, ϵ : $\widetilde{W}_{(k,B+\epsilon)}(T) - \widetilde{W}_{(k,B)}(T) \leq \widetilde{W}_{(k,B+\epsilon)}(S) - \widetilde{W}_{(k,B)}(S)$.*

We will prove this lemma for sufficiently small ϵ , which allows us to couple the executions tightly; the inequalities can then be added to imply the lemma for arbitrary ϵ .

By comparing the solutions to the linear program (1), we can ensure that any constraint that becomes tight in the solution for set T with bound $B + \epsilon$, but is not tight with bound B (and similarly for S) would not have become tight for any $\epsilon' < \epsilon$. This will localize the changes, and use revenue indifference for the seller. By summing over all such iterations (there will only be finitely many, because ϵ is chosen so that at least one more constraint becomes tight), we eventually prove the lemma.

In the analysis, we are interested in four different signaling schemes, constructed by CONSTRUCT-SIGNALING-SCHEME when run with different sets of price points and upper bounds B . We will assume here that $k \in S \subseteq T$. Specifically we define:

- $X = (x_{i,\sigma})_{\sigma,i}$: the probability mass of type i assigned to signal σ when the algorithm is run with price point set S and an upper bound of B .
- $X^+ = (x_{i,\sigma}^+)_{\sigma,i}$: probability mass for price point set S and an upper bound of $B + \epsilon$.
- $\widehat{X} = (\widehat{x}_{i,\sigma})_{\sigma,i}$: probability mass for price point set T and an upper bound of B .
- $\widehat{X}^+ = (\widehat{x}_{i,\sigma}^+)_{\sigma,i}$: probability mass for price point set T and an upper bound of $B + \epsilon$.

To avoid notational confusion, we will use σ to denote signals under the signaling scheme for $S \cup \{k\}$ and k_σ to denote their price points; signals under the signaling scheme for $T \cup \{k\}$ are denoted by ω , and their price points by \widehat{k}_ω .

Since we are interested in the change in the signaling scheme as we increase the bounds from B to $B + \epsilon$, we define $\delta_i^\sigma = x_{i,\sigma} - x_{i,\sigma}^+$ and $\widehat{\delta}_i^\omega = \widehat{x}_{i,\omega} - \widehat{x}_{i,\omega}^+$.

In order to understand the δ_i^σ better, consider the effect of changing the total probability constraint for the signal with price point k from B to $B + \epsilon$. When the signal with price point k is constructed, we may now add some more probability mass for types $i \geq k$. Subsequently, by Lemma 5.3, the algorithm continues with less (or equal) residual probability mass for all types. This might in turn mean that for later signals σ with price points $k' < k$, the probability mass for some type i might get used up earlier. In turn, this will speed up the addition of probability mass for types $i' \in \{k' + 1, \dots, i - 1\}$. In a sense, what happens is that the additional probability

mass for the signal with price point k “displaces” some of that mass from other signals, increasing different probability masses.

Consider some signal $\sigma \in Q$. Let $Q_{\leq \sigma} = \{\sigma' \in Q \mid \sigma' \leq \sigma\}$ be the set of signals constructed up to σ . We are interested in which price points may change their overall allocation of probability in the signals constructed up to σ as a result of the increase from B to $B + \epsilon$. Notice that the candidates are only those that did not have their probability mass already used up when the construction reached σ with an upper bound of B . To capture this, we define $k_\sigma < e_1^\sigma < e_2^\sigma < \dots < e_{m_\sigma}^\sigma$ to be the indices which still had probability mass available after signal σ was constructed, i.e., such that $x_{e_j^\sigma, Q_{\leq \sigma}} < p_{e_j^\sigma}$. For notational convenience, we define $e_{m_\sigma+1}^\sigma = n+1$ and $v_{n+1} = \infty$. Similarly, define $\hat{k}_\omega < \hat{e}_1^\omega < \hat{e}_2^\omega < \dots < \hat{e}_{\hat{m}_\omega}^\omega$ to be the indices with $\hat{x}_{\hat{e}_j^\omega, \hat{Q}_{\leq \omega}} < p_{\hat{e}_j^\omega}$.

We are particularly interested in indices whose overall total probability mass (at the end of CONSTRUCT-SIGNALING-SCHEME) can increase. For ease of notation, we therefore define $\hat{\sigma}_S = \max\{\sigma \mid k_\sigma \in S\}$, $\hat{\omega}_T = \max\{\omega \mid \hat{k}_\omega \in T\}$, and $e_j = e_j^{\hat{\sigma}_S}$, $\hat{e}_j = \hat{e}_j^{\hat{\omega}_T}$; note that this implies $x_{e_j, Q} < p_i$ for all j , and similarly for \hat{e}_j . One type $i < k$ could also see an increase, namely, if the displaced probability by an increase for a type $i' > k$ which eventually became saturated causes an increase in probability mass for some later signal with target $k' < k$. The only such target types would be $e_0 = \max\{i \leq k \mid x_{i, Q} < p_i\}$ and $\hat{e}_0 = \max\{i \leq k \mid \hat{x}_{i, \hat{Q}} < p_i\}$, respectively.

Notice that by making ϵ small enough, we can ensure that $x_{e_j^\sigma, Q_{\leq \sigma}}^+ < p_{e_j^\sigma}$ and $\hat{x}_{\hat{e}_j^\omega, \hat{Q}_{\leq \omega}}^+ < p_{\hat{e}_j^\omega}$ for all indices e_j^σ , \hat{e}_j^ω and signals σ, ω .

Define ϵ to be the supremum of all such values. This choice of ϵ ensures that at least one of the above inequalities becomes tight, but for any $\epsilon' < \epsilon$, we have $x_{e_j^\sigma, Q_{\leq \sigma}}^+ < p_{e_j^\sigma}$ and $\hat{x}_{\hat{e}_j^\omega, \hat{Q}_{\leq \omega}}^+ < p_{\hat{e}_j^\omega}$. With the chosen ϵ , for the index (or indices) where the inequality becomes tight, we still have a tight revenue constraint, because in the execution of CONSTRUCT-ONE-SIGNAL, the index was removed from I at the same time as k_σ , in the last round of the iteration.

By choosing the proper ϵ , at least one of e_j^σ and \hat{e}_j^ω is removed from the index set for the next larger value $B' = B + \epsilon$. Because there are only finitely many candidate indices, finitely many updates will reach the B such that $\hat{x}_{k, \omega^k} = p_k$ when running with a bound of B . Beyond that value of B , the signal ω^k cannot be further raised.

As we argued above, the increase in probability mass for the signal with price index k will reduce the probability mass available for other signals constructed subsequently in the algorithm CONSTRUCT-SIGNALING-SCHEME. Let σ be such a signal, with price point $k_\sigma < k$. A lack of available probability mass for low-value buyer types when σ is constructed may make its target price v_{k_σ} less attractive to the seller; to compensate, the signal must reduce the amount of probability mass for high-value buyers it uses. The following lemma captures the necessary reduction.

Lemma B.1 *Fix some signal σ , and let $i < i', j < j'$ be indices such that each of i, i', j, j' is either equal to k' or to one of the $e_{j''}^\sigma$. Then,*

$$\frac{\delta_{[j, j']}^\sigma}{\frac{1}{v_j} - \frac{1}{v_{j'}}} = \frac{\delta_{[i, i']}^\sigma}{\frac{1}{v_i} - \frac{1}{v_{i'}}}. \quad (11)$$

An analogous characterization holds for the $\hat{e}_{j''}^\omega$ and $\hat{\delta}_{[j, j']}^\omega$.

PROOF. Let $k' = k_\sigma$. Because $x_{e_j^\sigma, Q_{\leq \sigma}} < p_{e_j^\sigma}$, we can apply Lemma 4.1 to the iteration in which σ was constructed, and infer that for all $j \leq m_\sigma$,

$$v_{k'} \cdot x_{[k', n], \sigma} = v_{e_j^\sigma} \cdot x_{[e_j^\sigma, n], \sigma}, \quad v_{k'} x_{[k', n], \sigma}^+ = v_{e_j^\sigma} \cdot x_{[e_j^\sigma, n], \sigma}^+$$

whence $v_{k'} \cdot \delta_{[k', n]}^\sigma = v_{e_j^\sigma} \cdot \delta_{[e_j^\sigma, n]}^\sigma$ follows.

$$\delta_{[e_j^\sigma, e_{j'}^\sigma]}^\sigma = \delta_{[e_j^\sigma, n]}^\sigma - \delta_{[e_{j'}^\sigma, n]}^\sigma = \frac{v_{k'}}{v_{e_j^\sigma}} \cdot \delta_{[k', n]}^\sigma - \frac{v_{k'}}{v_{e_{j'}^\sigma}} \cdot \delta_{[k', n]}^\sigma = \left(\frac{1}{v_{e_j^\sigma}} - \frac{1}{v_{e_{j'}^\sigma}} \right) \cdot v_{k'} \cdot \delta_{[k', n]}^\sigma$$

Hence, $\frac{\delta_{[e_j^\sigma, e_{j'}^\sigma]}^\sigma}{\frac{1}{v_{e_j^\sigma}} - \frac{1}{v_{e_{j'}^\sigma}}} = v_{k'} \cdot \delta_{[k', n+1]}^\sigma$ for all $j < j' \leq m_\sigma + 1$, and it is easy to see that this calculation applies for $j = k'$ as well. \blacksquare

Lemma B.2 *The $e_j, \hat{e}_j, e_j^\sigma, \hat{e}_j^\omega$ satisfy the following subsequence properties.*

1. Each index \hat{e}_j also appears as an element in the sequence of e_j .
2. Let k' be a price point and σ, ω signals such that σ has price point k' under X and ω has price point k' under \hat{X} . Then, each index \hat{e}_j^ω also appears as an element in the sequence of e_j^σ .
3. If $\sigma > \sigma'$, then each index $e_j^\sigma \geq k_{\sigma'}$ also appears as an element in the sequence of $e_{j'}^{\sigma'}$. Similarly for \hat{e}_j^ω and $\hat{e}_{j'}^{\omega'}$.

PROOF. Let Q_S, Q_T denote the sets of signals corresponding to the price point sets S, T , respectively.

1. By Lemma 5.3, applied to $S \subseteq T$, we get that $x_{i, Q_S} \leq \hat{x}_{i, Q_T}$, so whenever $\hat{x}_{i, Q_T} < p_\sigma$, we also have $x_{i, Q_S} < p_\sigma$.
2. Analogous to the first part after applying Lemma 5.3 with $Q = Q_{S \leq \sigma}$ and $\hat{Q} = Q_{T \leq \omega}$. The corresponding price point sets are $\{k_\sigma \in S \mid k_\sigma \geq k'\} \subseteq \{\hat{k}_\omega \in T \mid \hat{k}_\omega \geq k'\}$, respectively.
3. Analogous to the first part after applying Lemma 5.3 with $Q = Q_{S \leq \sigma'}$ and $Q' = Q_{S \leq \sigma}$. \blacksquare

Lemma B.3 *Let $\sigma' > \sigma^k$ be a signal constructed after the one with price point k . Then, the total probability mass under signal σ' for the “initial saturated segment” $[k_{\sigma'}, e_1^{\sigma'}]$ cannot increase when B is raised to $B + \epsilon$. That is, $\delta_{[k_{\sigma'}, e_1^{\sigma'}]}^{\sigma'} \geq 0$. If furthermore, $e_1^{\sigma'} \leq k$, then $\delta_{[k_{\sigma'}, e_1^{\sigma'}]}^{\sigma'} = 0$. An analogous statement holds for $\hat{\delta}_{[k_{\omega'}, \hat{e}_1^{\omega'}]}^{\omega'}$ in place of $\delta_{[k_{\sigma'}, e_1^{\sigma'}]}^{\sigma'}$.*

PROOF. By definition of $e_1^{\sigma'}$, all the probability mass for indices $i \in [k_{\sigma'}, e_1^{\sigma'}]$ is used up by signals $1, \dots, \sigma'$ under both X and X^+ . Hence,

$$\sum_{\sigma \leq \sigma'} x_{[k_{\sigma'}, e_1^{\sigma'}], \sigma} = p_{[k_{\sigma'}, e_1^{\sigma'}]} = \sum_{\sigma \leq \sigma'} x_{[k_{\sigma'}, e_1^{\sigma'}], \sigma}^+.$$

Taking the difference and solving gives us that $\delta_{[k_{\sigma'}, e_1^{\sigma'}]}^{\sigma'} = -\sum_{\sigma < \sigma'} \delta_{[k_{\sigma'}, e_1^{\sigma'}]}^{\sigma} \geq 0$ by monotonicity (Lemma 5.3).

We prove the second part of the lemma by induction over $\sigma' > \sigma^k$. For the base case, let σ' be the least σ such that $e_1^{\sigma} \leq k$. Consider a signal $\sigma < \sigma'$. If $k_{\sigma} \leq e_1^{\sigma'}$, then part 3 of Lemma B.2 would imply that $e_1^{\sigma'}$ also appears as a e_j^{σ} for some j ; in particular, it would imply that $e_1^{\sigma} \leq e_1^{\sigma'} \leq k$, contradicting the minimality of σ' . Hence, $k_{\sigma} > e_1^{\sigma'}$, meaning that no probability mass is allocated to signals $\sigma < \sigma'$ for types $[k_{\sigma}, e_1^{\sigma'}]$. Because by definition, all probability mass for such types is used up by signals up to and including signal σ' , we conclude that $x_{[k_{\sigma'}, e_1^{\sigma'}], \sigma'} = p_{[k_{\sigma'}, e_1^{\sigma'}]} = x_{[k_{\sigma'}, e_1^{\sigma'}], \sigma'}^+$, and therefore $\delta_{[k_{\sigma'}, e_1^{\sigma'}]}^{\sigma'} = 0$.

For the induction step, we use our result from the first part that $\delta_{[k_{\sigma'}, e_1^{\sigma'}]}^{\sigma'} = -\sum_{\sigma < \sigma'} \delta_{[k_{\sigma'}, e_1^{\sigma'}]}^{\sigma}$. We will show that $\delta_{[k_{\sigma'}, e_1^{\sigma'}]}^{\sigma} = 0$ for all $\sigma < \sigma'$. First, when $k_{\sigma} > e_1^{\sigma'}$, we get that $x_{[k_{\sigma'}, e_1^{\sigma'}], \sigma} = x_{[k_{\sigma'}, e_1^{\sigma'}], \sigma}^+ = 0$, implying that $\delta_{[k_{\sigma'}, e_1^{\sigma'}]}^{\sigma} = 0$. Otherwise, when $k_{\sigma} \leq e_1^{\sigma'}$, we first observe that $x_{[k_{\sigma'}, k_{\sigma}], \sigma} = x_{[k_{\sigma'}, k_{\sigma}], \sigma}^+ = 0$, so $\delta_{[k_{\sigma'}, e_1^{\sigma'}]}^{\sigma} = \delta_{[k_{\sigma'}, e_1^{\sigma'}]}^{\sigma}$. Lemma B.2 implies that $e_1^{\sigma} \leq e_1^{\sigma'}$, so we can apply Lemma B.1 and the induction hypothesis to show that

$$\delta_{[k_{\sigma'}, e_1^{\sigma'}]}^{\sigma} \stackrel{\text{Lemma B.1}}{=} \delta_{[k_{\sigma'}, e_1^{\sigma'}]}^{\sigma} \cdot \left(\frac{1}{v_{k_{\sigma}}} - \frac{1}{v_{e_1^{\sigma'}}} \right) / \left(\frac{1}{v_{k_{\sigma}}} - \frac{1}{v_{e_1^{\sigma}}} \right) \stackrel{\text{I.H.}}{=} 0,$$

completing the proof. ■

We next get to the key lemma for submodularity. It compares the effects of the increase of ϵ under S and T on the same signal σ . At a very high level, it says that the effects on σ in terms of the allocated probability mass of low-value types is more severe for T than for S . The precise form is a bit subtle. To avoid notational confusion, we will use σ to denote signals under the signaling scheme for $S \cup \{k\}$ and k_{σ} to denote their price points; signals under the signaling scheme for $T \cup \{k\}$ are denoted by ω , and their price points by \hat{k}_{ω} . Recall that signals are sorted by decreasing price points, so that $k_{\sigma+1} < k_{\sigma}$, $\hat{k}_{\omega+1} < \hat{k}_{\omega}$. We frequently want to find, for a given signal ω , the signal σ with closest greater (or equal) price point to \hat{k}_{ω} . Hence, for any signal ω under $T \cup \{k\}$, we define $\lfloor \omega \rfloor = \max\{\sigma \mid k_{\sigma} \geq \hat{k}_{\omega}\}$.

Lemma B.4 *Let σ^k be the signal with price point k for $S \cup \{k\}$, and ω^k the signal with price point k for $T \cup \{k\}$. Let $\omega' \geq \omega^k$ be any signal, with price point $k' = \hat{k}_{\omega'}$. Then,*

$$\sum_{\sigma=\sigma^k}^{\lfloor \omega' \rfloor} \delta_{[k_{\sigma}, \hat{e}_j^{\omega'}]}^{\sigma} \leq \sum_{\omega=\omega^k}^{\omega'} \hat{\delta}_{[\hat{k}_{\omega}, \hat{e}_j^{\omega'}]}^{\omega}.$$

PROOF. We prove the statement by induction on ω' . The base case $\omega' = \omega^k$ is true because Lemma B.1, applied with $i = j = k$, $i' = \hat{e}_j^{\omega^k}$ and $j' = n + 1$ implies that $\delta_{[k, \hat{e}_j^{\omega^k}]}^{\sigma^k} = \hat{\delta}_{[k, \hat{e}_j^{\omega^k}]}^{\omega^k} = -\epsilon v_k \cdot \left(\frac{1}{v_k} - \frac{1}{v_{\hat{e}_j^{\omega^k}}} \right)$. We now focus on the induction step, and distinguish three cases.

1. If $\hat{k}_{\omega'} \notin S$, then $\sum_{\sigma=\sigma^k}^{\lfloor \omega' \rfloor} \delta_{[k_{\sigma}, \hat{e}_j^{\omega'}]}^{\sigma} = \sum_{\sigma=\sigma^k}^{\lfloor \omega' \rfloor - 1} \delta_{[k_{\sigma}, \hat{e}_j^{\omega'}]}^{\sigma}$. By induction hypothesis, the latter is at most $\sum_{\omega=\omega^k}^{\omega' - 1} \hat{\delta}_{[\hat{k}_{\omega}, \hat{e}_j^{\omega'}]}^{\omega}$. By Lemmas B.1 and B.3, $\hat{\delta}_{[\hat{k}_{\omega'}, \hat{e}_j^{\omega'}]}^{\omega'} \geq 0$, and adding this inequality proves the induction step.

2. If $\widehat{k}_{\omega'} \in S$ and $e_1^{\lfloor \omega' \rfloor} \leq k$, then write $\sigma' = \lfloor \omega' \rfloor$. By Lemma B.2, $\widehat{e}_j^{\omega'} = e_j^{\sigma'}$ for some j' , so we can apply Lemma B.1 and Lemma B.3 to conclude that $\delta_{[k_{\sigma'}, \widehat{e}_j^{\omega'}]}^{\sigma'} = \delta_{[k_{\sigma'}, e_1^{\sigma'}]}^{\sigma'} \cdot (\frac{1}{v_{k_{\sigma'}}} - \frac{1}{v_{\widehat{e}_j^{\omega'}}}) / (\frac{1}{v_{k_{\sigma'}}} - \frac{1}{v_{e_1^{\sigma'}}}) = 0$. As in the previous case, we get that $\widehat{\delta}_{[\widehat{k}_{\omega'}, \widehat{e}_j^{\omega'}]}^{\omega'} \geq 0$, and adding both terms to the inequality obtained from the induction hypothesis now completes the inductive step.
3. Otherwise, we are in the case that $\widehat{k}_{\omega'} \in S$ and $e_1^{\lfloor \omega' \rfloor} > k$; we again write $\sigma' = \lfloor \omega' \rfloor$. First, we are going to show that the lemma holds for σ' with $j = 1$. By Lemma 5.3, we obtain that

$$\sum_{\sigma=\sigma^k}^{\sigma'} \delta_{[k_{\sigma}, \widehat{e}_1^{\omega'}]}^{\sigma} = \sum_{\sigma=\sigma^k}^{\sigma'} (x_{[k_{\sigma}, \widehat{e}_1^{\omega'}], \sigma} - x_{[k_{\sigma}, \widehat{e}_1^{\omega'}], \sigma}^+) \leq 0,$$

whereas by definition of $\widehat{e}_1^{\omega'}$,

$$\sum_{\omega=\omega^k}^{\omega'} \widehat{\delta}_{[\widehat{k}_{\omega}, \widehat{e}_1^{\omega'}]}^{\omega} = \sum_{\omega=\omega^k}^{\omega'} (\widehat{x}_{[\widehat{k}_{\omega}, \widehat{e}_1^{\omega'}], \omega} - \widehat{x}_{[\widehat{k}_{\omega}, \widehat{e}_1^{\omega'}], \omega}^+) = p_{[\widehat{k}_{\omega}, \widehat{e}_1^{\omega'}]} - p_{[\widehat{k}_{\omega}, \widehat{e}_1^{\omega'}]} = 0.$$

Thus, we have shown that

$$\sum_{\sigma=\sigma^k}^{\sigma'} \delta_{[k_{\sigma}, \widehat{e}_1^{\omega'}]}^{\sigma} \leq \sum_{\omega=\omega^k}^{\omega'} \widehat{\delta}_{[\widehat{k}_{\omega}, \widehat{e}_1^{\omega'}]}^{\omega} = \sum_{\sigma=\sigma^k}^{\sigma'} \sum_{\omega: k_{\sigma} \leq \widehat{k}_{\omega} < k_{\sigma-1}, \widehat{k}_{\omega} \leq k} \widehat{\delta}_{[\widehat{k}_{\omega}, \widehat{e}_1^{\omega'}]}^{\omega}.$$

The induction hypothesis implies the same inequality with $\sigma'' < \sigma'$ in place of σ' , so that we have for all $\sigma'' \leq \sigma'$:

$$\sum_{\sigma=\sigma^k}^{\sigma''} \delta_{[k_{\sigma}, \widehat{e}_1^{\omega'}]}^{\sigma} \leq \sum_{\omega: k_{\sigma''} \leq \widehat{k}_{\omega} \leq k} \widehat{\delta}_{[\widehat{k}_{\omega}, \widehat{e}_1^{\omega'}]}^{\omega} = \sum_{\sigma=\sigma^k}^{\sigma''} \sum_{\omega: k_{\sigma} \leq \widehat{k}_{\omega} < k_{\sigma-1}, \widehat{k}_{\omega} \leq k} \widehat{\delta}_{[\widehat{k}_{\omega}, \widehat{e}_1^{\omega'}]}^{\omega}. \quad (12)$$

To extend the result to $j > 1$, consider any signal $\sigma \in \{\sigma^k + 1, \dots, \sigma'\}$. By Part (2) of Lemma B.2, $\widehat{e}_1^{\omega'}$ is equal to $e_j^{\sigma'}$ for some j . Because $e_1^{\sigma'} > k \geq k_{\sigma}$, by Part (3) of Lemma B.2, both $e_1^{\sigma'}$ and $\widehat{e}_1^{\omega'}$ occur as e_j^{σ} , $e_{j'}^{\sigma}$ for some j, j' . We are therefore allowed to apply Lemma B.1, and we can write

$$\begin{aligned} \sum_{\sigma=\sigma^k}^{\sigma'} \delta_{[k_{\sigma}, \widehat{e}_j^{\omega'}]}^{\sigma} &= \sum_{\sigma=\sigma^k}^{\sigma'} \delta_{[k_{\sigma}, \widehat{e}_1^{\omega'}]}^{\sigma} \cdot (\frac{1}{v_{k_{\sigma}}} - \frac{1}{v_{\widehat{e}_j^{\omega'}}}) / (\frac{1}{v_{k_{\sigma}}} - \frac{1}{v_{\widehat{e}_1^{\omega'}}}), \\ \sum_{\omega=\omega^k}^{\omega'} \widehat{\delta}_{[\widehat{k}_{\omega}, \widehat{e}_j^{\omega'}]}^{\omega} &= \sum_{\sigma=\sigma^k}^{\sigma'} \sum_{\omega: k_{\sigma} \leq \widehat{k}_{\omega} < k_{\sigma-1}, \widehat{k}_{\omega} \leq k} \widehat{\delta}_{[\widehat{k}_{\omega}, \widehat{e}_1^{\omega'}]}^{\omega} \cdot (\frac{1}{v_{\widehat{k}_{\omega}}} - \frac{1}{v_{\widehat{e}_j^{\omega'}}}) / (\frac{1}{v_{\widehat{k}_{\omega}}} - \frac{1}{v_{\widehat{e}_1^{\omega'}}}) \\ &\geq \sum_{\sigma=\sigma^k}^{\sigma'} (\frac{1}{v_{k_{\sigma}}} - \frac{1}{v_{\widehat{e}_j^{\omega'}}}) / (\frac{1}{v_{k_{\sigma}}} - \frac{1}{v_{\widehat{e}_1^{\omega'}}}) \cdot \sum_{\omega: k_{\sigma} \leq \widehat{k}_{\omega} < k_{\sigma-1}, \widehat{k}_{\omega} \leq k} \widehat{\delta}_{[\widehat{k}_{\omega}, \widehat{e}_1^{\omega'}]}^{\omega}. \end{aligned}$$

Because $\widehat{e}_j^{\omega'} \geq \widehat{e}_1^{\omega'}$, the function $x \mapsto (\frac{1}{x} - \frac{1}{v_{\widehat{e}_j^{\omega'}}}) / (\frac{1}{x} - \frac{1}{v_{\widehat{e}_1^{\omega'}}})$ is increasing in x for $x < v_{\widehat{e}_1^{\omega'}}$. Furthermore, by Inequality (12), we have domination of all prefix sums, so we can apply

Lemma B.5 with $a_\sigma = \delta_{[k_\sigma, \hat{e}_1^{\omega'}]}^\sigma$, $b_\sigma = \sum_{\omega: k_\sigma \leq \hat{k}_\omega < k_{\sigma-1}, \hat{k}_\omega \leq k} \hat{\delta}_{[\hat{k}_\omega, \hat{e}_1^{\omega'}]}^\omega$, and $c_\sigma = (\frac{1}{v_{k_\sigma}} - \frac{1}{v_{\hat{e}_j^{\omega'}}}) / (\frac{1}{v_{k_\sigma}} - \frac{1}{v_{\hat{e}_1^{\omega'}}})$ to conclude that $\sum_{\sigma=\sigma^k}^{\sigma'} \delta_{[k_\sigma, \hat{e}_j^{\omega'}]}^\sigma \leq \sum_{\omega=\omega^k}^{\omega'} \hat{\delta}_{[\hat{k}_\omega, \hat{e}_j^{\omega'}]}^\omega$, completing the inductive step. \blacksquare

Lemma B.5 *Let a_1, \dots, a_n and b_1, \dots, b_n be any numbers such that for all indices $i \leq n$, the prefixes satisfy that $\sum_{j=1}^i a_j \leq \sum_{j=1}^i b_j$. Then, for any coefficients $c_1 \geq c_2 \geq \dots \geq c_n \geq 0$, we have that $\sum_{j=1}^n c_j a_j \leq \sum_{j=1}^n c_j b_j$.*

PROOF. By defining $c_{n+1} = 0$, we can write $c_j = \sum_{i=j}^n (c_i - c_{i+1})$. Now, we get that

$$\sum_{j=1}^n c_j a_j = \sum_{j=1}^n \sum_{i=j}^n a_j (c_i - c_{i+1}) = \sum_{i=1}^n (c_i - c_{i+1}) \sum_{j=1}^i a_j \leq \sum_{i=1}^n (c_i - c_{i+1}) \sum_{j=1}^i b_j = \sum_{j=1}^n c_j b_j.$$

The inequality followed by the assumption on the prefixes and the non-negativity of all the $c_i - c_{i+1}$ terms. \blacksquare

PROOF OF LEMMA 5.2. Let σ^k be the signal with price point k to which probability mass was added. This addition can only lead to an increase in social welfare via an increase in buyer types that were previously not allocated to any signal, meaning that we are interested only in types i with $\sum_\sigma x_{i,\sigma} < p_i$. In addition to the indices e_j defined earlier for $j \geq 0$, define $e_0 > e_{-1} > e_{-2} > \dots$ to be all of the indices $i < e_0$ with $x_{Q,\sigma} < p_i$.

We will first show that those types $i < e_0$ do not actually lead to any welfare changes under X . Thereto, consider some type e_j with $j < 0$. Under X , buyers of this type can only be allocated to a signal σ with $k_\sigma \leq e_j$. For all such signals σ , by definition, we have that $e_1^\sigma \leq e_j < e_0 \leq k$, so Lemma B.3 implies that $\delta_{[k_\sigma, e_1^\sigma]}^\sigma = 0$. By Part (3) of Lemma B.2, e_j and e_{j+1} also occur as $e_{j'}^\sigma$, so we can apply Lemma B.1 to conclude that $\delta_{[e_j, e_{j+1}]}^\sigma = 0$, and by summing obtain that for all $i < 0$,

$$\sum_{\sigma} \delta_{e_j}^\sigma \stackrel{(*)}{=} \sum_{\sigma} \delta_{[e_j, e_{j+1}]}^\sigma = \sum_{\sigma: k_\sigma \leq e_j} \delta_{[e_j, e_{j+1}]}^\sigma = 0$$

In the step labeled (*), we are using the fact that e_i is the only index in the range $[e_i, e_{i+1})$ at which the probability mass can increase, and the probability mass for no other type will decrease.

By Part 1 of Lemma B.2, the \hat{e}_j form a subsequence of the e_j . To compare the effects of changes in the welfare, we partition the e_j into the segments formed by the \hat{e}_j ; formally, we define $[e_j] = \max\{\hat{e}_{j'} \mid \hat{e}_{j'} \leq e_j\}$. Let j^* be the index j such that $e_{j^*} = \hat{e}_1$.

We can now express the change in social welfare when adding k to S as follows:

$$\begin{aligned}
\widetilde{W}(X) - \widetilde{W}(X^+) &= \sum_j v_{e_j} \sum_{\sigma \geq \sigma^k} \delta_{e_j}^\sigma \\
&= \sum_{j \geq j^*} v_{e_j} \cdot \sum_{\sigma \geq \sigma^k} \delta_{[e_j, e_{j+1}]}^\sigma + \sum_{j=0}^{j^*-1} v_{e_i} \sum_{\sigma \geq \sigma^k} \delta_{e_j}^\sigma \\
&\stackrel{(*)}{=} \sum_{j \geq j^*} v_{e_j} \cdot \sum_{\sigma \geq \sigma^k} \delta_{[k_\sigma, e_1]}^\sigma \cdot \left(\frac{1}{v_{e_j}} - \frac{1}{v_{e_{j+1}}} \right) / \left(\frac{1}{v_{k_\sigma}} - \frac{1}{v_{e_1}} \right) + \sum_{j=0}^{j^*-1} v_{e_i} \sum_{\sigma \geq \sigma^k} \delta_{e_j}^\sigma \\
&= \left(\sum_{j \geq j^*} v_{e_j} \cdot \left(\frac{1}{v_{e_j}} - \frac{1}{v_{e_{j+1}}} \right) \right) \cdot \left(\sum_{\sigma \geq \sigma^k} \delta_{[k_\sigma, e_1]}^\sigma / \left(\frac{1}{v_{k_\sigma}} - \frac{1}{v_{e_1}} \right) \right) + \sum_{j=0}^{j^*-1} v_{e_i} \sum_{\sigma \geq \sigma^k} \delta_{e_j}^\sigma. \quad (13)
\end{aligned}$$

In the step labeled (*), we applied Lemma B.1. We were allowed to do so, because $e_{j+1} \geq e_j \geq \widehat{e}_1 \geq e_1 \geq k \geq k_\sigma$, allowing us to apply Part (3) of Lemma B.2.

We first analyze the second term $\sum_{j=0}^{j^*-1} v_{e_i} \sum_{\sigma \geq \sigma^k} \delta_{e_j}^\sigma$. If \widehat{e}_0 is defined, then

$$\sum_{j=0}^{j^*-1} v_{e_j} \sum_{\sigma \geq \sigma^k} \delta_{e_j}^\sigma \leq v_{\widehat{e}_0} \cdot \sum_{\sigma \geq \sigma^k} \sum_{j=1}^{j^*-1} \delta_{e_j}^\sigma \stackrel{\text{Lemma B.4}}{=} v_{\widehat{e}_0} \cdot \sum_{\sigma \geq \sigma^k} \delta_{[k_\sigma, \widehat{e}_1]}^\sigma \leq v_{\widehat{e}_0} \cdot \sum_{\omega \geq \omega^k} \widehat{\delta}_{[\widehat{k}_\omega, \widehat{e}_1]}^\omega.$$

Otherwise (\widehat{e}_0 is not defined), we will simply use the bound that $\sum_{\sigma \geq \sigma^k} \delta_{[k_\sigma, e_1]}^\sigma \leq 0$ by Lemma 5.3. For notational convenience, in this case, we will write $v_{\widehat{e}_0} = 0$.

We next consider the first factor of the first term of Equation (13). Because $e_j \geq \lfloor e_j \rfloor$, we obtain that

$$\begin{aligned}
\sum_{j \geq j^*} v_{e_j} \cdot \left(\frac{1}{v_{e_j}} - \frac{1}{v_{e_{j+1}}} \right) &\geq \sum_{j \geq j^*} v_{\lfloor e_j \rfloor} \cdot \left(\frac{1}{v_{e_j}} - \frac{1}{v_{e_{j+1}}} \right) = \sum_{j' \geq 1} v_{\widehat{e}_{j'}} \cdot \sum_{j: \lfloor e_j \rfloor = \widehat{e}_{j'}} \left(\frac{1}{v_{e_j}} - \frac{1}{v_{e_{j+1}}} \right) \\
&= \sum_{j' \geq 1} v_{\widehat{e}_{j'}} \cdot \left(\frac{1}{v_{\widehat{e}_{j'}}} - \frac{1}{v_{\widehat{e}_{j'+1}}} \right).
\end{aligned}$$

To analyze the second factor of the first term in Equation (13), we first apply Lemma B.1 to rewrite $\sum_{\sigma \geq \sigma^k} \delta_{[k_\sigma, e_1]}^\sigma / \left(\frac{1}{v_{k_\sigma}} - \frac{1}{v_{e_1}} \right) = \sum_{\sigma \geq \sigma^k} \delta_{[k_\sigma, \widehat{e}_1]}^\sigma / \left(\frac{1}{v_{k_\sigma}} - \frac{1}{v_{e_1}} \right)$. We also write

$$\begin{aligned}
\sum_{\omega \geq \omega^k} \widehat{\delta}_{[\widehat{k}_\omega, \widehat{e}_1]}^\omega / \left(\frac{1}{v_{\widehat{k}_\omega}} - \frac{1}{v_{\widehat{e}_1}} \right) &= \sum_{\sigma \geq \sigma^k} \sum_{\omega: k_\sigma \leq \widehat{k}_\omega < k_{\sigma-1}, \widehat{k}_\omega \leq k} \widehat{\delta}_{[\widehat{k}_\omega, \widehat{e}_1]}^\omega / \left(\frac{1}{v_{\widehat{k}_\omega}} - \frac{1}{v_{\widehat{e}_1}} \right) \\
&\geq \sum_{\sigma \geq \sigma^k} 1 / \left(\frac{1}{v_{k_\sigma}} - \frac{1}{v_{\widehat{e}_1}} \right) \sum_{\omega: k_\sigma \leq \widehat{k}_\omega < k_{\sigma-1}, \widehat{k}_\omega \leq k} \widehat{\delta}_{[\widehat{k}_\omega, \widehat{e}_1]}^\omega.
\end{aligned}$$

By Lemma B.4, we have that $\sum_{\sigma=\sigma^k}^{\lfloor \omega' \rfloor} \delta_{[k_\sigma, \widehat{e}_j^{\omega'}]}^\sigma \leq \sum_{\omega=\omega^k}^{\omega'} \widehat{\delta}_{[\widehat{k}_\omega, \widehat{e}_j^{\omega'}]}^\omega$ for all $\omega' \geq \omega^k$.

We can therefore apply Lemma B.5 with $a_\sigma = \delta_{[k_\sigma, \widehat{e}_1^{\omega'}]}^\sigma$, $b_\sigma = \sum_{\omega: k_\sigma \leq \widehat{k}_\omega < k_{\sigma-1}, \widehat{k}_\omega \leq k} \widehat{\delta}_{[\widehat{k}_\omega, \widehat{e}_1]}^\omega$, and

$c_\sigma = 1/(\frac{1}{v_{k_\sigma}} - \frac{1}{v_{\widehat{e}_1}})$, and conclude that

$$\begin{aligned} \sum_{\sigma \geq \sigma^k} \delta_{[k_\sigma, \widehat{e}_1]}^\sigma / (\frac{1}{v_{k_\sigma}} - \frac{1}{v_{\widehat{e}_1}}) &= \sum_{\sigma \geq \sigma^k} \delta_{[k_\sigma, \widehat{e}_1]}^\sigma / (\frac{1}{v_{k_\sigma}} - \frac{1}{v_{\widehat{e}_1}}) \leq \sum_{\sigma \geq \sigma^k} 1/(\frac{1}{v_{k_\sigma}} - \frac{1}{v_{\widehat{e}_1}}) \cdot \sum_{\omega: k_\sigma \leq \widehat{k}_\omega < k_{\sigma-1}, \widehat{k}_\omega \leq k} \widehat{\delta}_{[\widehat{k}_\omega, \widehat{e}_1]}^\omega \\ &\leq \sum_{\omega \geq \omega^k} \widehat{\delta}_{[\widehat{k}_\omega, \widehat{e}_1]}^\omega / (\frac{1}{v_{\widehat{k}_\omega}} - \frac{1}{v_{\widehat{e}_1}}). \end{aligned}$$

Recalling that $\sum_{\sigma \geq \sigma^k} \delta_{[k_\sigma, \widehat{e}_1]}^\sigma / (\frac{1}{v_{k_\sigma}} - \frac{1}{v_{\widehat{e}_1}}) \leq 0$ by Lemma 5.3, we now put all of these inequalities together, yielding that

$$\begin{aligned} \widetilde{W}(X) - \widetilde{W}(X^+) &\leq \left(\sum_{j' \geq 1} v_{\widehat{e}_{j'}} \cdot \left(\frac{1}{v_{\widehat{e}_{j'}}} - \frac{1}{v_{\widehat{e}_{j'+1}}} \right) \right) \cdot \left(\sum_{\omega \geq \omega^k} \widehat{\delta}_{[\widehat{k}_\omega, \widehat{e}_1]}^\omega / \left(\frac{1}{v_{\widehat{k}_\omega}} - \frac{1}{v_{\widehat{e}_1}} \right) \right) + v_{\widehat{e}_0} \cdot \sum_{\omega \geq \omega^k} \widehat{\delta}_{[\widehat{k}_\omega, \widehat{e}_1]}^\omega \\ &= \sum_{j' \geq 1} v_{\widehat{e}_{j'}} \cdot \sum_{\omega \geq \omega^k} \widehat{\delta}_{[\widehat{k}_\omega, \widehat{e}_1]}^\omega \cdot \left(\frac{1}{v_{\widehat{e}_{j'}}} - \frac{1}{v_{\widehat{e}_{j'+1}}} \right) / \left(\frac{1}{v_{\widehat{k}_\omega}} - \frac{1}{v_{\widehat{e}_1}} \right) + v_{\widehat{e}_0} \cdot \sum_{\omega \geq \omega^k} \widehat{\delta}_{[\widehat{k}_\omega, \widehat{e}_1]}^\omega \\ &\stackrel{\text{Lemma B.1}}{=} \sum_{j' \geq 1} v_{\widehat{e}_{j'}} \cdot \sum_{\omega \geq \omega^k} \widehat{\delta}_{[\widehat{e}_{j'}, \widehat{e}_{j'+1}]}^\omega + v_{\widehat{e}_0} \cdot \sum_{\omega \geq \omega^k} \widehat{\delta}_{[\widehat{k}_\omega, \widehat{e}_1]}^\omega \\ &= \sum_{j' \geq 0} v_{\widehat{e}_{j'}} \cdot \sum_{\omega \geq \omega^k} \widehat{\delta}_{\widehat{e}_{j'}}^\omega \\ &= \widetilde{W}(\widehat{X}) - \widetilde{W}(\widehat{X}^+). \end{aligned}$$

Finally, by noticing that the respective increases in sanitized welfare are the negatives of the terms here (i.e., $\widetilde{W}(X^+) - \widetilde{W}(X)$ and $\widetilde{W}(\widehat{X}^+) - \widetilde{W}(\widehat{X})$, respectively), we complete the proof of Lemma 5.2. \blacksquare

A note on the approximation with a single signal for power law distribution

June 28, 2016

Suppose the value of the buyer v is drawn from a distribution D such that $\Pr_{v \sim D}[v \geq x] = x^{-p}$, with $0 < p < 1$. Assume the support of distribution D is $[n]$, then we have

$$\begin{aligned}\Pr[v = x] &= x^{-p} - (x+1)^{-p} \approx px^{-p-1} \\ \Pr[v = n] &= n^{-p}.\end{aligned}$$

The social welfare from D is

$$\begin{aligned}\sum_{x=1}^n \Pr[v = x]x &\approx \sum_{x=1}^{n-1} px^{-p} + n^{1-p} \\ &\approx \int_1^n px^{-p} dx + n^{1-p} \\ &= \frac{p(n^{1-p} - 1)}{1-p} + n^{1-p} \\ &= \frac{n^{1-p} - p}{1-p} = \frac{n^{1-p} - 1}{1-p} + 1.\end{aligned}$$

Consider a single signal of price x_0 , the probability mass of type x allocated to the signal is

$$\begin{aligned}q_x &= \Pr[v = x \text{ and signal is sent}] \\ &= \Pr[v = x_0] / \left(\frac{1}{x_0} - \frac{1}{x_0+1}\right) \left(\frac{1}{x} - \frac{1}{x+1}\right) \\ q_n &= \Pr[v = x_0] / \left(\frac{1}{x_0} - \frac{1}{x_0+1}\right) (1/n)\end{aligned}$$

The social welfare obtained from the signal with price x_0 is

$$\begin{aligned}
\sum_{x=x_0}^n xq_x &= \mathbf{Pr}[v = x_0] / \left(\frac{1}{x_0} - \frac{1}{x_0 + 1} \right) \left(1 + \sum_{x=x_0}^{n-1} x \left(\frac{1}{x} - \frac{1}{x+1} \right) \right) \\
&= r \left(1 + \sum_{x=x_0}^{n-1} \frac{1}{x+1} \right) \\
&\approx r(1 + \log n - \log x_0) \\
&\approx px_0^{1-p} (1 + \log n - \log x_0)
\end{aligned}$$

where $r = \mathbf{Pr}[v = x_0] / \left(\frac{1}{x_0} - \frac{1}{x_0 + 1} \right)$

Now consider the the social welfare of the signal x as a function $g(x) = px^{1-p}(1 + \log n - \log x)$. The maximum of this function is obtained when $\log x = \log n - \frac{1}{1-p} + 1$. When $p < 1 - \frac{1}{\log n}$, we have $x \geq 1$, so the signal is well defined, with a social welfare of

$$\frac{pn^{1-p}}{e^p(1-p)}$$

which is a $\frac{p}{e^p}$ approximation of the total social welfare.

Otherwise, when $p > 1 - \frac{1}{\log n}$, the best signal price is 1, which results in a social welfare of $2\mathbf{Pr}[v = 1](1 + \sum_{x=1}^{n-1} \frac{1}{x+1}) \approx 2\mathbf{Pr}[v = 1] \log n$.

Assume $p = 1 - f(n)$, with $\lim_{n \rightarrow \infty} f(n) = 0$. We are going to characterise the increasing speed of $\frac{n^{1-p}-1}{1-p}$ is actually in the same order of $\log n$, so the signal of price 1 gives constant approximation of the total social welfare.

Consider the limit $\lim_{n \rightarrow \infty} \frac{n^{f(n)} - 1}{f(n) \log n}$, which is $\lim_{n \rightarrow \infty} n^{f(n)}$. Notice that when $\lim_{n \rightarrow \infty} f(n) \log n$ is finite, the limit $\lim_{n \rightarrow \infty} n^{f(n)} = \lim_{n \rightarrow \infty} e^{f(n) \log n}$ is finite. So for $f(x) \in \frac{1}{\Omega(\log n)}$, the social welfare of price 1 is a constant approximation for the total social welfare.